**Math 21b - Fall 1998 Final Exam**

1) TRUE/FALSE

(a) If $\mathbf{A}$ is a $3 \times 3$ orthogonal matrix, then det($\mathbf{A}$) is an eigenvalue of $\mathbf{A}$.

(b) If $T$ is a linear transformation from $\mathbf{R}^n$ to $\mathbf{R}^n$ which sends orthogonal vectors to orthogonal vectors, then $T$ is an orthogonal transformation.

(c) If $\mathbf{A}$ is a real $4 \times 4$ matrix with determinant 1/2 and with no real eigenvalue, then $\mathbf{A}^n \to \mathbf{0}$ as $n \to \infty$

(d) Suppose that $\mathbf{A}$ is a $2 \times 2$ real matrix and that the discrete dynamical system $\mathbf{x}(t+1) = \mathbf{A}\mathbf{x}(t)$ has a non-constant solution satisfying $\mathbf{x}(t+4) = \mathbf{x}(t)$, then $\mathbf{0}$ is an unstable equilibrium solution of $\mathbf{x}(t+1) = \mathbf{A}\mathbf{x}(t)$.

(e) If there are orthonormal eigenbases of $\mathbf{R}^n$ for each of two matrices $\mathbf{A}$ and $\mathbf{B}$, then there is also one for $\mathbf{A} + \mathbf{B}$.

(f) The solutions to $\dfrac{d^2 f}{dt^2} + \dfrac{df}{dt} + f = e^t$ form a two dimensional linear subspace of $C^\infty$.

2) (a) Find the eigenvalues and eigenvectors for $\begin{bmatrix} 3 & 2 \\ 10 & 4 \end{bmatrix}$.

(b) Find a matrix $\mathbf{A}$ such that $\mathbf{A}^3 = \begin{bmatrix} 3 & 2 \\ 10 & 4 \end{bmatrix}$.

3) Suppose that $\mathbf{A}$ is a $3 \times 5$ matrix and $\mathbf{B}$ is a $5 \times 3$ matrix such that $\mathbf{AB} = \mathbf{I}_3$.

(a) Show that the non-zero vectors in Ker($\mathbf{A}$) are eigenvectors for $\mathbf{BA}$. What is their eigenvalue?

(b) Show that the non-zero vectors in Im($\mathbf{B}$) are eigenvectors for $\mathbf{BA}$. What is their eigenvalue?

(c) List all the eigenvalues of $\mathbf{BA}$ with their geometric multiplicities. Is there an eigenbasis of $\mathbf{R}^5$ for $\mathbf{BA}$?

4) Let $\mathbf{v}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$. Match the following matrices with the verbal descriptions below.

(a) $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ (b) $\mathbf{B} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ (c) $\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$ (d) $\mathbf{D} = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix}$

(i) The matrix, with respect to the standard basis of $\mathbf{R}^3$, for reflection in the line spanned by $\mathbf{v}_1$.

(ii) The matrix with respect to the standard basis of $\mathbf{R}^3$, for projection onto the plane spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$.

(iii) The matrix, with respect to the basis $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ of $\mathbf{R}^3$, for reflection in the plane spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$.

(iv) The matrix, with respect to the basis $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ of $\mathbf{R}^3$, for the rotation about the line spanned by $\mathbf{v}_1$ which takes $\mathbf{v}_2$ to $\mathbf{v}_3$.

5) Let $E \subset \mathbf{R}^2$ denote the image of the unit circle (i.e. the set of $\mathbf{y} \in \mathbf{R}^2$ with $\mathbf{y}^{\mathrm{T}}\mathbf{y} = 1$) under the linear transformation given by $\begin{bmatrix} 1 & 3/2 \\ 0 & 1 \end{bmatrix}$.

(a) Show that $E$ is the set of vectors $\mathbf{x} \in \mathbf{R}^2$ such that $\mathbf{x}^{\mathrm{T}}\begin{bmatrix} 4 & -6 \\ -6 & 13 \end{bmatrix}\mathbf{x} = 4$.

(b) Find an orthonormal eigenbasis $\mathbf{v}$, $\mathbf{w}$ of $\mathbf{R}^2$ for $\begin{bmatrix} 4 & -6 \\ -6 & 13 \end{bmatrix}$.

(c) Show that there are two real numbers $\lambda$, $\mu$, which you should identify explicitly, such that $E$ is the set of vectors $a\mathbf{v} + b\mathbf{w}$ with $\lambda a^2 + \mu b^2 = 4$.

(d) What is the nearest that any point of $E$ comes to $\mathbf{0}$?

6) Consider the continuous dynamical system $\left\{ \begin{aligned} \frac{dx}{dt} &= y \\ \frac{dy}{dt} &= (1 - x^2)y - x \end{aligned} \right\}$.

(a) Sketch the nullclines for this linear system in the region $-4 \le x, y \le +4$, identify the equilibrium point and indicate the approximate direction of the trajectories in each region of your diagram. DO NOT mark any trajectories.

(b) Linearize these equations near the equilibrium solution. Is this a stable or unstable equilibrium? Describe briefly and qualitatively the behavior of the trajectories near this equilibrium solution.

(c) Now, sketch a possible phase portrait (system of trajectories) for the original dynamical system in the region $-4 \le x, y \le +4$.

7) Consider the linear transformation $T : C^\infty \to C^\infty$ given by $T(f) = \frac{df}{dt} - f$.

(a) Show that $e^t \int_0^t e^{-s} g(s)\, ds$ is a solution to the differential equation $\frac{df}{dt} - f = g$.

(b) What is the image and what is the kernel of $T$?

(c) What is the image and what is the kernel of $T^2$?

(d) Find all solutions of the equation $\frac{d^2 f}{dt^2} - 2\frac{df}{dt} + f = e^{-t}$.

8) (a) Find the Fourier series for the function $f : [-\pi, \pi] \to \mathbf{R}$ given by $f(t) = t$.

(b) Solve the heat equation $\frac{\partial T(x,t)}{\partial t} = \frac{\partial^2 T(x,t)}{\partial x^2}$ in the region $t \ge 0$, $0 \le x \le \pi$ and subject to the initial conditions

$$T(x, 0) = x \quad \text{for } 0 < x < \pi$$

$$T(0, t) = T(\pi, t) = 0 \quad \text{for } t > 0.$$