1. (a) FALSE

(b) TRUE

(c) FALSE — $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}3\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}$

(d) FALSE — if A is 3×6

$$\text{rank}(A) + \dim(\ker A) = 6$$
$$\text{rank} \leq 3, \text{ so } \dim(\ker A) \geq 3$$

(e) TRUE — $\begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}$

2. $V_1 = \text{Span}\left(\begin{bmatrix}0\\1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\\1\end{bmatrix}\right) = \left\{\begin{bmatrix}a\\b\\b\\a\end{bmatrix}\right\}$

$W_4 = \left\{\begin{bmatrix}a\\b\\c\\d\end{bmatrix} : a=d, b=c\right\}$ so $V_1 = W_4$

$V_2 = \left\{\begin{bmatrix}a\\a+b\\a+b\\b\end{bmatrix}\right\}$ $W_5 = \left\{\begin{bmatrix}a\\b\\b\\b-a\end{bmatrix}\right\}$ $V_2 = W_5$

$V_3 = \left\{\begin{bmatrix}a\\b\\c\\d\end{bmatrix} : \begin{matrix}b=c\\c=0\end{matrix}\right\}$ $W_1 = \left\{\begin{bmatrix}a\\0\\0\\b\end{bmatrix}\right\}$ $V_3 = W_1$

$V_4 = \left\{\begin{bmatrix}a\\b\\c\\d\end{bmatrix} : \begin{matrix}a+b+c=0\\b+c+d=0\end{matrix}\right\}$ $W_2 = \left\{\begin{bmatrix}a\\-a-b\\b\\a\end{bmatrix}\right\}$ $V_4 = W_2$

$V_5 = \left\{\begin{bmatrix}a\\a\\b\\b\end{bmatrix}\right\}$ $W_3 = \left\{\begin{bmatrix}a\\b\\c\\d\end{bmatrix} : \begin{matrix}a=b\\c=d\end{matrix}\right\}$ $V_5 = W_3$

③ (a) $\left[\begin{array}{ccc|ccc} 0 & 1 & 3 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & 5 & 0 & 0 & 1 \end{array}\right]$

$\left[\begin{array}{ccc|ccc} -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 3 & 1 & 0 & 0 \\ 0 & 2 & 5 & 0 & 0 & 1 \end{array}\right]$ switch II and I

$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & -1 \end{array}\right] \begin{array}{l} -1 \cdot \mathrm{I} \\ \\ 2 \cdot \mathrm{II} - \mathrm{III} \end{array}$

$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -5 & 0 & 3 \\ 0 & 0 & 1 & 2 & 0 & -1 \end{array}\right] \begin{array}{l} \\ \mathrm{II} - 3 \cdot \mathrm{III} \\ \\ \end{array}$

Inverse $= \begin{bmatrix} 0 & -1 & 0 \\ -5 & 0 & 3 \\ 2 & 0 & -1 \end{bmatrix}$

(b) $A^2 = \begin{bmatrix} -1 & 6 & 15 \\ 0 & -1 & -3 \\ -2 & 10 & 25 \end{bmatrix}$

4 (a) If A is a shear parallel to L
(b)

1) $A\vec{v} = \vec{v}$ if $\vec{v} \in L \Rightarrow (A - I_2)\vec{v} = \vec{0}$

2) $A\vec{v} - \vec{v}$ is on L for all $\vec{v}$

By 1),

$$(A - I_2)\vec{v} = \vec{0} \text{ if } \vec{v} \text{ is on } L$$

in other words, L is a subset of $\ker(A - I_2)$

By 2),

$$(A - I_2)\vec{v} \text{ is on } L \text{ for all } \vec{v}$$

in other words, $\operatorname{Im}(A - I_2)$ is a subset of L

We know that $\operatorname{Im}(A - I_2) \neq \{\vec{0}\}$, so
we must have $\operatorname{Im}(A - I_2) = L$ (part b)

We also know that $\ker(A - I_2) \neq \mathbb{R}^2$, so
we must have $\ker(A - I_2) = L$ (part a)

(c) $(A - I_2)^2 \vec{v} = (A - I_2)\vec{x}$, where $\vec{x} = (A - I_2)\vec{v}$

$\vec{x}$ is on L, by (b)

so $(A - I_2)\vec{x} = \vec{0}$, by (a)

So $(A - I_2)^2 = 0$

(5) (a) rank $(A) = 2$

One possibility for A is $\begin{bmatrix} 3 & -1 & 0 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$

(b) rank $(B) = 1$

One possibility for B is $\begin{bmatrix} -2 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(c) NO

Ker (AB) contains Ker (B): if $\vec{v} \in \ker B$, $(AB)\vec{v} = A(B\vec{v}) = A(\vec{0}) = \vec{0}$, so $\vec{v} \in \ker(AB)$ as well.

So $\dim(\ker AB) \geq 2$
which means rank $(AB) \leq 1$ (rank + nullity $= 3$)