Math 21b Midterm 1 Solutions - Spring 2001

1. True or False (no explanation is necessary):

   (a) FALSE - A system of linear equations always has 0, 1, or infinitely many solutions.

   (b) FALSE - Consider the following 2 x 2 matrix A:

   $$\begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix}.$$

   Notice that $A^2 = 0$, but $A \neq 0$.

   (c) TRUE - Consider the following 2 x 2 matrix B:

   $$\begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}$$

   and notice that $B^2 = -I_2$.

   (d) TRUE - Recall homework problem 44a of Section 3.1.

   (e) TRUE - Observe:

   $$\frac{-4}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \frac{2}{3}\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

   Also, these vectors are in $\mathbb{R}^3$, and the most linearly independent vectors you can have in $\mathbb{R}^3$ is 3.

2. True or False (no explanation is necessary):

   (a) TRUE - The linear transformation whose matrix representation is $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ works.

   (b) FALSE - Interchanging vertices A and B while leaving vertices C and D fixed is not a linear transformation.

   (c) FALSE - The origin $(0, 0)$ must be fixed by a linear transformation, but this case does not fix the origin.

   (d) TRUE - Suppose the corners of the square $ABCD$ are at $(\pm 1, \pm 1)$. Then the linear transformation whose matrix representation is $\begin{bmatrix} 0 & 1 \\ -.5 & .5 \end{bmatrix}$ works.

(e) TRUE - The linear transformation whose matrix representation is $\begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix}$ works.

3. (a) $A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 3 & 2 \\ 1 & -2 & 0 & 0 \\ 2 & 0 & 4 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 3 & 2 \\ 0 & 1 & 1 & 1 \\ 1 & -2 & 0 & 0 \\ 2 & 0 & 4 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 3 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & -3 & -3 & -2 \\ 2 & 0 & 4 & 1 \end{bmatrix}$

$\longrightarrow \begin{bmatrix} 1 & 1 & 3 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & -3 & -3 & -2 \\ 0 & -2 & -2 & -3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 3 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & -2 & -2 & -3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 3 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$

$\longrightarrow \begin{bmatrix} 1 & 1 & 3 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = rref(A).$

(b) From the above matrix we see that $x_1 + 2x_3 = x_2 + x_3 = x_4 = 0$. Thus a basis for $ker(A)$ is the vector $\begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix}$.

(c) Since the first, second, and fourth columns of $rref(A)$ have the leading 1's, a basis for the image of A is $\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$.

(d) Since A sends every vector in $ker(A)$ to 0, the nonzero matrix $B = \begin{bmatrix} -2 & -2 & -2 & -2 \\ -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, each of whose columns is the vector found in (b), will be such that $AB = 0$:

$$AB = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 3 & 2 \\ 1 & -2 & 0 & 0 \\ 2 & 0 & 4 & 1 \end{bmatrix} \begin{bmatrix} -2 & -2 & -2 & -2 \\ -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

4. $x + ky = 0$; $x + k^2 y = k^2$.

(a) Putting this system into a matrix yields $A = \begin{bmatrix} 1 & k & 0 \\ 1 & k^2 & k^2 \end{bmatrix}$, and subtracting the first row from the second gives $\begin{bmatrix} 1 & k & 0 \\ 0 & k^2 - k & k^2 \end{bmatrix}$.

Now, if $k^2 - k \neq 0$, we can reduce this further to $\begin{bmatrix} 1 & k & 0 \\ 0 & 1 & \frac{k}{k-1} \end{bmatrix}$ and finally to $\begin{bmatrix} 1 & 0 & \frac{-k^2}{k-1} \\ 0 & 1 & \frac{k}{k-1} \end{bmatrix}$.

Thus, given that $k(k-1) \neq 0$, there is a unique solution to this system.

Consider now when $k(k-1) = 0$. If $k = 0$, then the matrix A is really $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ which has infinitely many solutions. If $k = 1$, the matrix A is $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ which has no solution. For all other values of k, there is a unique solution to the system.

(b) We are projecting onto the line $x + 2y = 0$, so we need a direction vector for this line. One such vector is $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$. This vector has length $\sqrt{5}$, so we must actually use the vector $\vec{u} = \begin{bmatrix} \frac{2\sqrt{5}}{5} \\ \frac{-\sqrt{5}}{5} \end{bmatrix}$. Recall that the orthogonal projection of a vector $\vec{v}$ onto this line is found by computing $(\vec{u} \cdot \vec{v})\vec{u}$. Thus the answer to this problem is:

$$\left(\begin{bmatrix} \frac{2\sqrt{5}}{5} \\ \frac{-\sqrt{5}}{5} \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -4 \end{bmatrix}\right) \begin{bmatrix} \frac{2\sqrt{5}}{5} \\ \frac{-\sqrt{5}}{5} \end{bmatrix} = 2\sqrt{5} \begin{bmatrix} \frac{2\sqrt{5}}{5} \\ \frac{-\sqrt{5}}{5} \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}.$$

5. A is a shear parallel to the line L and $A \neq I_2$.

(a) A shear is defined to be such that $T(\vec{v}) = \vec{v}$ for all vectors $\vec{v}$ in L and $T(\vec{x}) - \vec{x}$ is in L for all vectors $\vec{x}$ in the plane. The kernel of $A - I_2$ is the set of all vectors $\vec{v}$ such that $A(\vec{v}) = \vec{v}$. Thus the kernel of $A - I_2$ is the line L, since $A \neq I_2$.

(b) By the second property listed in (a), the image of $A - I_2$ is also the line L.

(c) Since, by (b), the image of $A - I_2$ is the line L, the image of $(A - I_2)^2$ is the image of the line L under $A - I_2$. By (a), $A - I_2$ sends the line L to 0. Thus $(A - I_2)^2$ is the zero transformation.