Math 21b Spring '97 Exam 1

1. True or false?

   (a) If S and A are invertible $n \times n$ matrices, then
   $$(SAS^{-1})^{-1} = S^{-1}A^{-1}S.$$

   (b) The vectors
   $$\begin{bmatrix} 1 \\ -1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} -4 \\ 5 \\ 1 \\ 0 \end{bmatrix}, \text{ and } \begin{bmatrix} -2 \\ 3 \\ 1 \\ 6 \end{bmatrix}$$
   are linearly independent.

   (c) If the coefficient matrix of a system of m equations in n variables has rank less than m, the system has either no solutions or infinitely many solutions.

   (d) A linear transformation from R^5 to R^3 has a kernel of dimension at least 2.

   (e) If A is an $n \times n$ matrix such that $A^2 + 2A - I_n = 0$, then A is invertible.

2. Find a basis of the kernel of A and a basis of the image of A, where
$$A = \begin{bmatrix} 2 & -2 & 0 & 4 & -4 \\ -1 & 1 & 0 & -2 & 3 \\ 0 & 0 & 1 & 1 & -3 \\ 0 & 0 & 2 & 2 & 1 \end{bmatrix}.$$

3. For which choices of the constant k is the matrix
$$\begin{bmatrix} k^2 & k & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$
invertible?

4. Find three 2×2 matrices A, B, and C, each representing a shear parallel to either the x-axis or the y-axis, such that ABC represents a counterclockwise rotation through 90^o. Draw the images of the unit square (with corners at (0,0); (1,0); (1,1); and (0,1)) under the transformations C, BC, and ABC.

5. Let $A = \begin{bmatrix} \sqrt{6} & -\sqrt{2} \\ \sqrt{2} & \sqrt{6} \end{bmatrix}$.

   (a) The linear transformation $T(\vec{x}) = A\vec{x}$ is a (check one and fill in the blanks):
   __ rotation through an angle of _____
   __ rotation-dilation with rotation angle _____and dilation factor _____
   __ reflection through the line _____
   __ orthogonal projection onto the line _____
   __ shear parallel to the line _____

   (b) What is A^{-1}?

   (c) What is A^{10}? You may write your answer in any form that does not involve powers or products of matrices.

   (d) What is A^k, where k is a positive integer? You may write your answer in any form that does not involve powers or products of matrices.

   (e) What is the smallest positive integer k such that $A^k = \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix}$ for some numbers c_1 and c_2.