Math 21b Spring '97 Exam 1 Solutions

1. (a) False:
$$(SAS^{-1})^{-1} = \left(S^{-1}\right)^{-1} A^{-1}S^{-1} = SA^{-1}S^{-1} \neq S^{-1}A^{-1}S$$
in general.
   (b) False: The third vector is twice the first plus the second.
   (c) False: The system $1x_1 = 7, 2x_1 = 14$ is a counterexample.
   (d) True: $\dim(\text{Im}) \leq 3$ and $\dim(\text{Ker}) = 5 - \dim(\text{Im})$.
   (e) True: The equation says $A(A+2) = I$, so $A^{-1} = A + 2$.

2. Reduce to rref and solve for Ker and identify pivot columns to get a basis for Im:
$$\text{Basis of Ker}(A) = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}. \qquad \text{Basis of Im}(A) = \left\{ \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -4 \\ 3 \\ -3 \\ 1 \end{bmatrix} \right\}.$$

3. Doing Gauss-Jordan reduces A to $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & (1-k^2) \end{bmatrix}$. So A is invertible if and only if $1 - k^2 \neq 0$, which is equivalent to $k \neq \pm 1$.

4. The matrix of a shear parallel to the x-axis is of the form $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$, and a shear parallel to the y-axis is of the form $\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$. (Think about the transformations of $\vec{e}_1$ and $\vec{e}_2$ to determine these matrices.)
We will try to write
$$(*) \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} (**)$$
(It's not wise to use two shears of the same type consecutively, which would result in another shear of the same type.) Multiplying out and solving (*) gives $b = 1$ and $a = c = -1$. Multiplying out and solving (**) gives $b = -1$ and $a = c = 1$. Either solution is fine; we choose the first, which may be written
$$A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \; B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \; C = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}.$$
Graphically, C maps the original unit square to a parallelogram with vertices (0,0); (1,0); (0,1); and (-1,1). Then B maps this parallelogram to a parallelogram with vertices (0,0); (1,1); (0,1); and (-1,0). That parallelogram is mapped by A to the square with vertices (0,0); (0,1); (-1,1); and (-1,0).

5. (a) Notice A is in the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. This is a rotation/dilation by counterclockwise angle $\tan^{-1}(b/a) = \pi/6$ and factor $\sqrt{a^2+b^2} = \sqrt{8}$.
   (b) A^{-1} is a rotation/dilation by a clockwise angle $\pi/6$ and factor $1/\sqrt{8}$.
   (c) A^{10} is a rotation/dilation by a counterclockwise angle $10\pi/6$ and factor $\sqrt{8}^{10} = 8^5$.
   (d) A^{10} is a rotation/dilation by a counterclockwise angle $k\pi/6$ and factor $\sqrt{8}^k$.
   (e) $k = 6$ gives rotation by π and dilation by $\sqrt{8}^3$. The matrix for this is $\begin{bmatrix} -8^3 & 0 \\ 0 & -8^3 \end{bmatrix}$.