FALL 96 2nd Exam Solutions

① DATA gives inconsistent system
$$\begin{cases} a-b+c=-2\\ a-b+c=-1\\ a \qquad\quad =0\\ a+b+c=-1\\ a+b+c=-2\end{cases} \Rightarrow \begin{bmatrix}1&-1&1\\1&-1&1\\1&0&0\\1&1&1\\1&1&1\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix}=\begin{bmatrix}-2\\-1\\0\\-1\\-2\end{bmatrix}$$
$$\Rightarrow A\vec{c}=\vec{y}$$

Solutions given by normal EQUATION $A^TA\vec{c}=A^T\vec{y}$

calculate $A^TA=\begin{bmatrix}5&0&4\\0&4&0\\4&0&4\end{bmatrix}$ $A^T\vec{y}=\begin{bmatrix}-6\\0\\-6\end{bmatrix}$ Solve $\left[\begin{array}{ccc|c}5&0&4&-6\\0&4&0&0\\4&0&4&-6\end{array}\right]\to\left[\begin{array}{ccc|c}1&0&0&0\\0&1&0&0\\0&0&1&-3/2\end{array}\right]$

So $f(x)=0+0x-\frac{3}{2}x^2=-\frac{3}{2}x^2$

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②ⓐ expand along 2nd column
$$\det A=2\det\begin{bmatrix}2&1&0&3\\-4&2&-1&0\\3&-3&0&0\\-3&1&0&0\end{bmatrix}=2\cdot3\det\begin{bmatrix}-4&2&-1\\3&-3&0\\-3&1&0\end{bmatrix}=2\cdot3\cdot(-1)\left[3-9\right]=(-6)(-6)=\boxed{36}$$

②b subtract 1st row from all other rows → same determinant.
$$\Rightarrow\left[\begin{array}{c|cccc}a&b&b&\cdots&b\\ \hline 0&a-b&0&\cdots&0\\0&a-b&a-b&&0\\0&a-b&a-b&\ddots&0\\0&a-b&a-b&&0\\0&a-b&a-b&&0\\0&a-b&a-b&&0\\0&a-b&a-b&\cdots&(a-b)\end{array}\right]=a\cdot(a-b)^7$$
since 7×7 MATRIX IS LOWER TRIANGULAR, det = prod. of its diagonal entries.

If $a=1, b=-1 \Rightarrow \det A=2^7\neq 0$
so A will be invertible

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③ⓐ $\|\vec{v}_1\|=\sqrt{6}$ ⓑ $\sqrt{\det A^TA}$ where $A=\begin{bmatrix}\vec{v}_1&\vec{v}_2\end{bmatrix}=\begin{bmatrix}1&0\\-1&-2\\0&1\\2&-1\end{bmatrix}$. $A^TA=\begin{bmatrix}6&0\\0&6\end{bmatrix}$

Area $=\sqrt{\det A^TA}=\sqrt{36}=6$

ⓒ $\vec{w}_1=\frac{\vec{v}_1}{\|\vec{v}_1\|}=\frac{1}{\sqrt6}\begin{bmatrix}1\\-1\\0\\2\end{bmatrix}$ $\vec{v}_2\cdot\vec{w}_1=0$, so $\vec{w}_2=\frac{\vec{v}_2}{\|\vec{v}_2\|}=\frac{1}{\sqrt6}\begin{bmatrix}0\\-2\\1\\-1\end{bmatrix}$

$\vec{v}_3\cdot\vec{w}_1=0$, $\vec{v}_3\cdot\vec{w}_2=\sqrt6$, so $\vec{v}_3-(\vec{v}_3\cdot\vec{w}_1)\vec{w}_1-(\vec{v}_3\cdot\vec{w}_2)\vec{w}_2=\begin{bmatrix}1\\1\\2\\0\end{bmatrix}\Rightarrow\vec{w}_3=\frac{1}{\sqrt6}\begin{bmatrix}1\\1\\2\\0\end{bmatrix}$

ⓓ $BB^T=$ Projection onto span$\{\vec{w}_1,\vec{w}_2\}$ which is 2D in $\mathbb{R}^4$.
The evals are therefore $\{+1,+1,0,0\}$. $E_1=\text{Span}\{\vec{w}_1,\vec{w}_2\}=W$
$E_0=W^\perp$.

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④ $\vec{x}(t+1)=A\vec{x}(t) \Rightarrow \vec{x}(t)=A^t\vec{x}(0)$

ⓐ $\lambda I-A=\begin{bmatrix}\lambda-1&1\\-1&\lambda-1\end{bmatrix}$ Det $=\lambda^2-2\lambda+2=0 \Rightarrow \lambda=1\pm i$ eigenvalues

$\cancel{\lambda=1+i\Rightarrow\left[\begin{array}{cc|c}i&1&0\\-1&i&0\end{array}\right]\to\vec{w}=\begin{bmatrix}1\\-i\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}+i\begin{bmatrix}0\\-1\end{bmatrix}=\vec{u}+i\vec{v}\quad \mathcal{B}=\{\vec{v},\vec{u}\}}$

$\cancel{P=\begin{bmatrix}0&1\\-1&0\end{bmatrix}\quad P^{-1}=\begin{bmatrix}0&-1\\1&0\end{bmatrix}\quad P^{-1}AP=}$ A already in form $\begin{bmatrix}a&-b\\b&a\end{bmatrix}$,

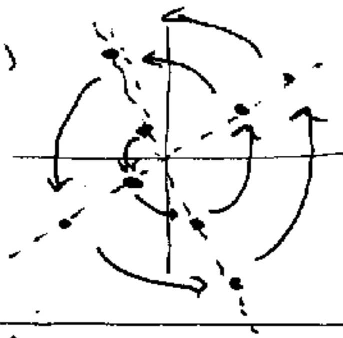
a ROTATION-DILATION MATRIX $A=\begin{bmatrix}1&-1\\1&1\end{bmatrix}=\sqrt2\,R_{\pi/4}$

where $R_{\pi/4}$ = ROTATION through $\pi/4$, counterclockwise.

next →

4 cont. $A^t = (\sqrt{2})^t R_{t\cdot\pi/4}$ $\quad \vec{x}(t) = (\sqrt{2})^t R_{t\cdot\pi/4}\, \vec{x}(0)$

(SPIRAL OUT)



d $\vec{x}(12) = (\sqrt{2})^{12} R_{3\pi} \begin{bmatrix} -.1 \\ +.2 \end{bmatrix} = 64 \begin{bmatrix} .1 \\ -.2 \end{bmatrix} = \begin{bmatrix} 6.4 \\ -12.8 \end{bmatrix}$

Trajectory does enter all four quadrants.

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5 a $A\vec{x}_0 = \vec{x}_1$

$A\vec{x}_1 = \vec{x}_2$

$A\vec{x}_2 = \vec{x}_3 = a\vec{x}_0 + b\vec{x}_1$

$\mathcal{B} = \{\vec{x}_0, \vec{x}_1, \vec{x}_2\} \Rightarrow [A]_{\mathcal{B}} = \begin{bmatrix} 0 & 0 & a \\ 1 & 0 & b \\ 0 & 1 & 0 \end{bmatrix}$

b $\vec{x}_0 = \begin{bmatrix} 0 \\ 0 \\ 1/2 \end{bmatrix}$ $\vec{x}_1 = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$ $\vec{x}_2 = \begin{bmatrix} 0 \\ -5 \\ 0 \end{bmatrix}$ $\Rightarrow P = \begin{bmatrix} 0 & 3 & 0 \\ 0 & 0 & -5 \\ 1/2 & 0 & 0 \end{bmatrix}$ $\quad P^{-1} = \begin{bmatrix} 0 & 0 & 2 \\ 1/3 & 0 & 0 \\ 0 & -1/5 & 0 \end{bmatrix}$

$[A]_{\mathcal{B}} = P^{-1}AP \Rightarrow A = P[A]_{\mathcal{B}}P^{-1}$

$\vec{x}_3 = \begin{bmatrix} 9 \\ 0 \\ 1 \end{bmatrix} = A\vec{x}_2$

$= a\vec{x}_0 + b\vec{x}_1 = a\begin{bmatrix} 0 \\ 0 \\ 1/2 \end{bmatrix} + b\begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3b \\ 0 \\ a/2 \end{bmatrix}$

$\Rightarrow \begin{matrix} 3b = 9 \\ a/2 = 1 \end{matrix} \Rightarrow \begin{matrix} b = 3 \\ a = 2 \end{matrix}$

$\Rightarrow [A]_{\mathcal{B}} = \begin{bmatrix} 0 & 0 & 2 \\ 1 & 0 & 3 \\ 0 & 1 & 0 \end{bmatrix}$

So $A = \begin{bmatrix} 0 & 3 & 0 \\ 0 & 0 & -5 \\ 1/2 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 2 \\ 1 & 0 & 3 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1/3 & 0 & 2 \\ 1/3 & 0 & 0 \\ 0 & -1/5 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -9/5 & 6 \\ -5/3 & 0 & 0 \\ 0 & -1/5 & 0 \end{bmatrix}$

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6 $\vec{x}(t+1) = A\vec{x}(t)$ $\quad A = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix}$ $\quad \lambda I - A = \begin{bmatrix} \lambda - .8 & -.3 \\ -.2 & \lambda - .7 \end{bmatrix}$

$\vec{x}(0) = \begin{bmatrix} 5000 \\ 0 \end{bmatrix}$

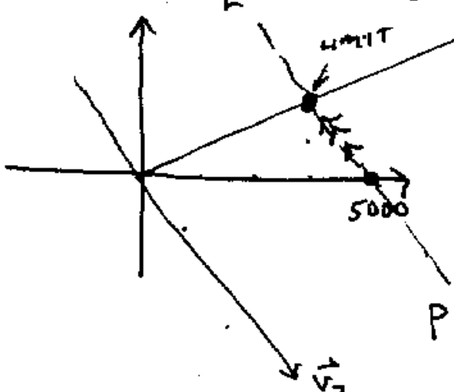
$\Rightarrow \lambda^2 - \frac{3}{2}\lambda + \frac{1}{2} = 0$

$(\lambda - 1)(\lambda - \frac{1}{2}) = 0$ $\quad \lambda_1 = 1,\ \lambda_2 = \frac{1}{2}$

$\lambda_1 = 1 \Rightarrow \left[\begin{array}{cc|c} .2 & -.3 & 0 \\ -.2 & .3 & 0 \end{array}\right] \to \vec{v}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

$\lambda_2 = \frac{1}{2} \Rightarrow \left[\begin{array}{cc|c} -.3 & -.3 & 0 \\ -.2 & -.2 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$\Rightarrow \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$



$\vec{x}(t) = c_1\lambda_1^t\vec{v}_1 + c_2\lambda_2^t\vec{v}_2 = c_1\vec{v}_1 + c_2(\frac{1}{2})^t\vec{v}_2$

$\lim_{t\to+\infty} \vec{x}(t) = c_1\vec{v}_1$

$P = \begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix}$ $\quad P^{-1} = +\frac{1}{5}\begin{bmatrix} +1 & +1 \\ +2 & -3 \end{bmatrix}$ $\quad \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = P^{-1}\begin{bmatrix} 5000 \\ 0 \end{bmatrix} = \frac{5000}{5}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1000 \\ 2000 \end{bmatrix}$

$c_1 = 1000$ $\quad \therefore\ \vec{x}(t) \to \begin{bmatrix} 3000 \\ 2000 \end{bmatrix} = \begin{bmatrix} \text{MEGAMART} \\ \text{BROADWAY} \end{bmatrix}$