Math 21b Second Midterms -Spring 1995 and Fall 1996

Math 21b Exam 2 - April 10, 1995

(All questions worth 20%)

1) Find the determinant of $\mathbf{A} = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{bmatrix}$.

2) Find the least squares solution of

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

3) Let $\mathbf{A} = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix}$.

a) Find the characteristic polynomial $f_{\mathbf{A}}(\lambda)$ of **A**.

b) Find the eigenvalues and corresponding eigenvectors of **A**.

c) Find $\mathbf{A}^{100} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

4) Let $\mathbf{M} = \begin{bmatrix} 2 & -3 \\ 1 & -1 \end{bmatrix}$.

a) Find the complex eigenvalues of **A**. Write your answers in polar form.

b) Determine the (asymptotic) stability of the dynamical system $\vec{x}(t+1) = \mathbf{M}\,\vec{x}(t)$. Describe the long term behavior of the system qualitatively.

c) Is there a positive k such that $\mathbf{M}^k = \mathbf{I}_2$? If so, find the smallest such k. If not, explain why not.

5) Consider three vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ in $\mathbf{R}^3$. In the table below, we list all the dot products $\mathbf{v}_i \cdot \mathbf{v}_j$.

$\cdot$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$
$\mathbf{v}_1$	4	0	0
$\mathbf{v}_2$	0	2	1
$\mathbf{v}_3$	0	1	5

Find the volume of the box (= parallelepiped) formed by $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

Math 21b Exam 2 - November 26, 1996

1) a) Find the quadratic function

$$f(x) = \mathrm{a} + \mathrm{b}x + \mathrm{c}x^2$$

that best fits the points (-1,-2), (-1,-1), (0,0), (1,-1), (1,-2) in the sense of least squares.

b) Does the graph of the function you found in part a) pass through the origin?

2a) Calculate the determinant of the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & 1 & 0 & 3 \\ 1 & 2 & 1 & 0 & 0 \\ -4 & 0 & 2 & -1 & 0 \\ 3 & 0 & -3 & 0 & 0 \\ -3 & 0 & 1 & 0 & 0 \end{bmatrix}$$

2b) Consider the 8×8 matrix

$$\mathbf{A} = \begin{bmatrix} a & b & b & b & b & b & b & b \\ a & a & b & b & b & b & b & b \\ a & a & a & b & b & b & b & b \\ a & a & a & a & b & b & b & b \\ a & a & a & a & a & b & b & b \\ a & a & a & a & a & a & b & b \\ a & a & a & a & a & a & a & b \\ a & a & a & a & a & a & a & a \end{bmatrix}.$$

(The i-th row has i a's, followed by $(8 - i)$ b's.)

i) Find det (**A**).

ii) If $a = 1$ and $b = -1$, is **A** invertible?

3) We are given three vectors in $\mathbf{R}^4$:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ -2 \\ 1 \\ -1 \end{bmatrix}, \text{ and } \mathbf{v}_3 = \begin{bmatrix} 1 \\ -1 \\ 3 \\ -1 \end{bmatrix}.$$

a) Find the length of $\mathbf{v}_1$.

b) Find the area of the parallelogram determined by the vectors $\{ \mathbf{v}_1, \mathbf{v}_2 \}$.

c) Using the vectors $\{ \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \}$ above, find an <u>orthonormal</u> basis $\{ \mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3 \}$ for the subspace which they span, using the Gram-Schmidt orthogonalization method.

d) Given the matrix $\mathbf{B} = [\mathbf{w}_1\ \ \mathbf{w}_2]$ where $\mathbf{w}_1$ and $\mathbf{w}_2$ are as found in the previous problem. Find the eigenvalues of the matrix $\mathbf{BB}^T$, and their algebraic and geometric multiplicities. Describe the eigenspaces. (Hint: <u>Do no further calculation! There is no need to find the characteristic polynomial</u>.)

4) Consider the dynamical system

$$\mathbf{x}(t+1) = \mathbf{A}\,\mathbf{x}(t) \text{ , with } \mathbf{A} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

a) Find the eigenvalues of **A**.

b) If $\mathbf{x}(0) = \begin{bmatrix} -0.1 \\ 0.2 \end{bmatrix}$, will the trajectory of $\mathbf{x}(t)$ enter all four quadrants?

c) What is $\mathbf{x}(12)$?

5) Consider a dynamical system $\mathbf{x}(t+1) = \mathbf{A}\,\mathbf{x}(t)$, where **A** is a 3×3 matrix. Suppose that $\mathbf{x}(0)$, $\mathbf{x}(1)$, and $\mathbf{x}(2)$ are linearly independent. Further suppose that $\mathbf{x}(3) = a\mathbf{x}(0) + b\mathbf{x}(1)$.

a) Find the matrix of **A** relative to the basis $B = \{\mathbf{x}(0), \mathbf{x}(1), \mathbf{x}(2)\}$, i.e. find $[\mathbf{A}]_B$.

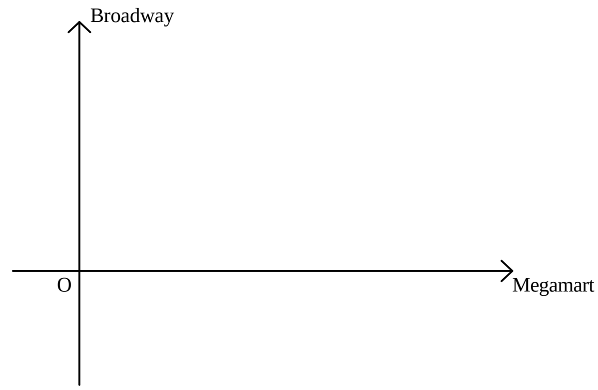
b) If we are further given that $\mathbf{x}(0) = \begin{bmatrix} 0 \\ 0 \\ 1/2 \end{bmatrix}$,

$\mathbf{x}(1) = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{x}(2) = \begin{bmatrix} 0 \\ -5 \\ 0 \end{bmatrix}$, and $\mathbf{x}(3) = \begin{bmatrix} 9 \\ 0 \\ 1 \end{bmatrix}$.

Determine the matrix **A** relative to the standard basis of $\mathbf{R}^3$.

6) Not long ago, the Broadway Marketplace opened a couple of blocks from here, selling fresh meat and produce and some groceries. For argument's sake, let's say that this store has one competitor, Megamart Inc., and that there is a constant total customer base of 5,000 people, all of whom originally went to Megamart. Weekly surveys show that Megamart retains 80% of it's customers from the previous week with the rest going to Broadway. Broadway keeps 70% of its customers from one week to the next, with the rest going for the cheaper prices at the Megamart.

a) Indicate, on the axes provided, the trajectory associated with this discrete dynamical system.



b) Will there come a point where the number of people going to each store will remain about the same from week to week? If so, how many will be going to each store? For full credit, you must justify all of your conclusions.