1. Let $B$ be the matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

(a) Find $\det(B)$.

(b) Find all eigenvalues (real and/or complex) of $B$ and their **algebraic** multiplicities.

(c) For each **real** eigenvalue of $B$, find the eigenspace corresponding to it and its geometric multiplicity. (Clearly indicate which eigenspaces and multiplicities belong to which eigenvalues.)

2. For each of the following, circle T for true or F for false. No explanation is necessary. Assume all the matrices A and B below are arbitrary $n \times n$ matrices with real entries.

(a) T F It is always the case that $\det(A^T A) \geq 0$.

(b) T F If $A$ and $B$ are symmetric, then $AB$ is symmetric.

(c) T F If $\vec{v}$ is an eigenvector of $A$ and is also an eigenvector of $B$, then $\vec{v}$ is an eigenvector of $AB$.

(d) T F A linear transformation $T : \mathbf{R}^n \longrightarrow \mathbf{R}^n$ is orthogonal if and only if $T(\vec{x}) \cdot T(\vec{y}) = \vec{x} \cdot \vec{y}$ for all vectors $\vec{x}$ and $\vec{y}$ in $\mathbf{R}^n$.

(e) T F If $C$ is any $m \times n$ matrix and $\vec{b}$ is any vector in $\mathbf{R}^m$, then the equation $C\vec{x} = \vec{b}$ always has a unique least squares solution.

3. Find the determinant of the matrix

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ -1 & 0 & 3 & 4 & 5 & 6 & 7 & 8 \\ -1 & -2 & 0 & 4 & 5 & 6 & 7 & 8 \\ -1 & -2 & -3 & 0 & 5 & 6 & 7 & 8 \\ -1 & -2 & -3 & -4 & 0 & 6 & 7 & 8 \\ -1 & -2 & -3 & -4 & -5 & 0 & 7 & 8 \\ -1 & -2 & -3 & -4 & -5 & -6 & 0 & 8 \\ -1 & -2 & -3 & -4 & -5 & -6 & -7 & 0 \end{bmatrix}.$$

4. (a) Consider the matrix

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 1 & -1 \\ 4 & 0 & 8 \\ -1 & 1 & 1 \\ -3 & 1 & -1 \end{bmatrix}.$$

   Use the Gram-Schmidt process to find an orthonormal basis for the image of $A$.

   (b) Find an orthogonal matrix $Q$ and an upper triangular matrix $R$ such that $A = QR$.

   (c) Show that for an arbitrary matrix $A$ (not just the particular $A$ used in parts a and b), the least squares solution of $A\vec{x} = \vec{b}$ simplifies to

$$\vec{x} = R^{-1}Q^T\vec{b}$$

   (assuming that $A^TA$ is invertible). Here $A = QR$ is a decomposition of the arbitrary matrix $A$ into a product of an orthogonal $Q$ and an upper triangular $R$.

5. Suppose 2500 TV watchers all watch exactly one rerun each night: *Star Trek* or *Melrose Place.* Assume 70% of those watching *Star Trek* on a given night watch it again the next night, and 80% of the *Melrose Place* viewers watch *Melrose Place* the next night. On April 8, 1998 there are 2024 *Star Trek* viewers.

   (a) How many viewers watch *Star Trek* on April 18, 1998?

   (b) It's a summer night early in the next century. How many viewers watch each show that night? (Round your answers to the nearest integer.)