Solutions for 2nd Midterm Math 21b Spring '99

① a) True – if $S\vec{v}=\lambda\vec{v}$ then $S(T\vec{v})=(ST)\vec{v}=(TS)\vec{v}=T(S\vec{v})=T(\lambda\vec{v})=\lambda(T\vec{v})$.

b) True – odd matrices always have at least one real eigenvalue; also complex eigenvalues always occur in complex conjugate pairs. So the eigenvalues are $3+i$, $3-i$ and one real one. All distinct, so diagonalizable.

c) False – counterexample $A=$ zero matrix.

d) True – can get from first matrix to second via 2 swaps – $\vec{v_1}$ with $\vec{v_2}$, then $\vec{v_1}$ with $\vec{v_4}$. Even # of swaps – same de

e) True – characteristic poly is $\lambda^2-(\text{tr}A)\lambda+(\det A)=0$. If $\det A<0$ then "b^2-4ac" >0 so 2 real distinct solutions.

f) True – if $\vec{0}$ is stable, the real parts of all eigenvalues are negative, so the trace must be negative (since $\text{tr}A=\lambda_1+\dots+\lambda_n$)

② a) Form $A=\begin{bmatrix}1&0\\2&1\\-1&2\\3&0\end{bmatrix}$. Area $=\sqrt{\det(A^TA)}=\sqrt{\det\begin{bmatrix}15&0\\0&5\end{bmatrix}}=\sqrt{75}$. [Or notice $\vec{u}\cdot\vec{v}=0$ so area $=\|\vec{u}\|\,\|\vec{v}\|$ since it's a rectangle

b) A is 5×5 and has 5 distinct real eigenvalues (0, 5, 9, 12, 14). So $\boxed{\text{YES}}$ A is diagonalizable.

c) $\begin{bmatrix}2&8&8&0\\0&3&0&0\\0&0&5&0\\0&0&0&7\end{bmatrix}$

d) i) $\left.\begin{array}{l}\text{Trace must be 8}\\ \text{Det must be 121}\end{array}\right\}$ because $f_A(\lambda)=\lambda^n-(\text{tr}A)\lambda^{n-1}+\dots+\det A$ <u>Ans</u>: C

ii) Trace must be 8
Det must be −72 <u>Ans</u>: E

$\text{tr}A=8$	$\text{tr}B=-8$	$\text{tr}C=8$	$\text{tr}D=20$	$\text{tr}E=8$
$\det A=0$ (2 columns the same)	$\det B=?$	$\det C=121$	$\det D=121$	$\det E=-72$

③ a) $A(-A)=I_n$ so A is invertible with $A^{-1}=-A$. [or $\det(A^2)=\det(-I_n)=(-1)^n\neq 0$ so $\det A\neq 0$]

b) $\det(A^2)=\det(-I_n)=(-1)^n$.
So $(\det A)^2=(-1)^n$.
But $(\det A)^2>0$ so n must be even.

c) Suppose λ is a real eigenvalue, with eigenvector $\vec{v}$, so $A\vec{v}=\lambda\vec{v}$.
Then $A^2\vec{v}=\lambda^2\vec{v}$. But $A^2=-I_n$ so $\lambda^2\vec{v}=-\vec{v}$, so $\lambda^2=-1$, impossible. So no real eigenvalues.

④ $A=$ standard matrix of $T=SBS^{-1}$ $\left(S=\begin{bmatrix}1&1\\2&3\end{bmatrix},\ B=\begin{bmatrix}1&-1\\1&1\end{bmatrix}\right)$
$=\begin{bmatrix}6&-2\\13&-4\end{bmatrix}$

Matrix of T with respect to $\begin{bmatrix}1\\3\end{bmatrix},\begin{bmatrix}1\\4\end{bmatrix}$ $=R^{-1}AR$, $\left(R=\begin{bmatrix}1&1\\3&4\end{bmatrix}\right)$
$=\begin{bmatrix}-1&-5\\1&3\end{bmatrix}$

⑤ a) $A=\begin{bmatrix}2&-1&1\\-1&2&-1\\1&-1&2\end{bmatrix}$

b) $f_A(\lambda)=(\lambda-1)^2(\lambda-4)$.

Eigenvalues	4	1
Alg. mult.	1	2
Geom mult.	1	2

$\ker(I_3-A)=\ker\begin{bmatrix}-1&+1&-1\\+1&-1&+1\\-1&+1&-1\end{bmatrix}=\ker\begin{bmatrix}1&-1&1\\0&0&0\\0&0&0\end{bmatrix}$. Basis for E_1: $\underbrace{\begin{bmatrix}1\\1\\0\end{bmatrix}}_{\vec{v_1}}$, [

$\ker(4I_3-A)=\ker\begin{bmatrix}2&1&-1\\1&2&+1\\-1&1&2\end{bmatrix}=\ker\begin{bmatrix}0&-3&-3\\1&2&1\\0&3&3\end{bmatrix}=\ker\begin{bmatrix}0&0&0\\1&0&-1\\0&1&1\end{bmatrix}$.

Basis for E_4: $\underbrace{\begin{bmatrix}1\\-1\\1\end{bmatrix}}_{\vec{v_3}}$

c) Yes, A is positive definite, because all its eigenvalues are positive.

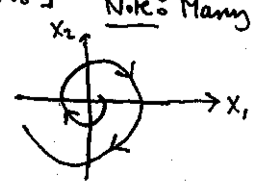
d) Orthonormal eigen basis $\vec{w_1},\vec{w_2},\vec{w_3}$, where

$\vec{w_1}=\frac{\vec{v_1}}{\|\vec{v_1}\|}=\begin{bmatrix}\frac{1}{\sqrt2}\\\frac{1}{\sqrt2}\\0\end{bmatrix}$, $\vec{w_2}=\frac{\vec{v_3}}{\|\vec{v_3}\|}=\begin{bmatrix}\frac{1}{\sqrt3}\\-\frac{1}{\sqrt3}\\\frac{1}{\sqrt3}\end{bmatrix}$, $\vec{w_3}=\vec{w_1}\times\vec{w_2}=\begin{bmatrix}1/\sqrt6\\-1/\sqrt6\\-2/\sqrt6\end{bmatrix}$ (Or use Gram-Schmidt)

Note: Many possible answers.

⑥ $A=\begin{bmatrix}a&-b\\b&a\end{bmatrix}$. Eigenvalues: $\lambda=a\pm ib$

a) $\left.\begin{array}{l}\text{Real part of the eigenvalues is positive } (a=1) \text{ so spirals out.}\\ \text{Spirals clockwise because b is negative (eg velocity vector at } \begin{bmatrix}1\\0\end{bmatrix} \text{ is } \begin{bmatrix}1\\-1\end{bmatrix}\text{)}\end{array}\right\}$



b) Spirals in so $a<0$
counterclockwise so $b>0$

Straight lines so $b=0$
Pointing away from origin so $a>0$

Spirals out so $a>0$
counterclockwise so $b>0$

Joins up so $a=0$
counterclockwise so $b>0$.