BRIEF SOLUTIONS

① T T F T T F F F T F

②
$$\begin{pmatrix} 1 & 0 & 3 & -1 \\ 2 & 1 & 5 & 1 \\ 2 & 0 & 3 & -1 \\ 1 & 1 & 2 & 2 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 3 & -1 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & -3 & 1 \\ 0 & 1 & -1 & 3 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 3 & -1 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & -1/3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 8/3 \\ 0 & 0 & 1 & -1/3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(a) $\ker A = \operatorname{span}\begin{pmatrix} 0 \\ -8 \\ 1 \\ 3 \end{pmatrix}$ has dimension 0

(b) $\operatorname{Im} A = \operatorname{span}\begin{pmatrix} 1 \\ 2 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 5 \\ 3 \\ 2 \end{pmatrix}$ has dimension 3

(c) $\det A = 0$

3 $\begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & t \\ 1 & 0 & 1 & t^2 \\ -1 & 0 & 1 & t^3 \end{pmatrix}$ $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1+t \\ 0 & -1 & 0 & t^2-1 \\ 0 & 1 & 2 & t^3+1 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 & -1 & -t \\ 0 & 1 & 2 & 1+t \\ 0 & 0 & 2 & t^2+t \\ 0 & 0 & 0 & t^3-t \end{pmatrix}$

(a) $\det A = 2(t^3-t)$

(b) $t \neq 0, \pm 1$: $\ker A = \{0\}$

$t=0$: $\ker A = \ker \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \ker \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \operatorname{span} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$

$t=1$: $\ker A = \ker \begin{pmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \ker \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \operatorname{span} \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$

$t=-1$: $\ker A = \ker \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \ker \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \operatorname{span} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$

4 (a) If $\lambda^2 - 4\lambda + 13 = 0$ when $\lambda = \dfrac{-4 \pm \sqrt{16-52}}{2} = -2 \pm \sqrt{4-13}$
$= -2 \pm 3i$

$\therefore y(x) = Ae^{-2x}\sin 3x + Be^{-2x}\cos 3x$

(b) $y(0) = B = 1$
$y'(0) = \left(-2Ae^{-2x}\sin 3x + 3Ae^{-2x}\cos 3x - 2Be^{-2x}\cos 3x - 3Be^{-2x}\sin 3x\right)\Big|_x$

$= 3A - 2B = 2$

$\therefore B = 1 \qquad 3A = 2 + 2B = 4 \quad \therefore A = 4/3$

$y(x) = 4/3\, e^{-2x}\sin 3x + e^{-2x}\cos 3x$

5 (a) A is diagonalisable as it is symmetric.
$f_A(\lambda) = \lambda^2 - 4\lambda + 3 = (\lambda-3)(\lambda-1)$

$E_1 = \ker \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \ker \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \operatorname{span} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$
$E_3 = \ker \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} = \ker \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} = \operatorname{span} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\therefore A \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$

(b) Yes $S = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$

6\. (a) $3/5$ of second year students graduate at the end of that year.

(b) $1/4,\ 1/5,\ 1$

(c) 600

7\. (a) $T(A+B) = (A+B)\begin{pmatrix}0&-1\\1&0\end{pmatrix} - \begin{pmatrix}0&-1\\1&0\end{pmatrix}(A+B)$

$$= A\begin{pmatrix}0&-1\\1&0\end{pmatrix} - \begin{pmatrix}0&-1\\1&0\end{pmatrix}A + B\begin{pmatrix}0&-1\\1&0\end{pmatrix} - \begin{pmatrix}0&-1\\1&0\end{pmatrix}B$$

$$= T(A) + T(B)$$

$T(\lambda A) = (\lambda A)\begin{pmatrix}0&-1\\1&0\end{pmatrix} - \begin{pmatrix}0&-1\\1&0\end{pmatrix}(\lambda A) = \lambda T(A)$

(b) $T\begin{pmatrix}a&b\\c&d\end{pmatrix} = \begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}0&-1\\1&0\end{pmatrix} - \begin{pmatrix}0&-1\\1&0\end{pmatrix}\begin{pmatrix}a&b\\c&d\end{pmatrix}$

$$= \begin{pmatrix}b&-a\\d&-c\end{pmatrix} - \begin{pmatrix}-c&-d\\a&b\end{pmatrix} = \begin{pmatrix}b+c&d-a\\d-a&-(b+c)\end{pmatrix}$$

T has matrix $\begin{pmatrix}0&1&1&0\\-1&0&0&1\\-1&0&0&1\\0&-1&-1&0\end{pmatrix}$.

(c) $\ker T = \left\{\begin{pmatrix}a&b\\-b&a\end{pmatrix}\right\}$ has basis $\left\{\begin{pmatrix}1&0\\0&1\end{pmatrix}, \begin{pmatrix}0&1\\-1&0\end{pmatrix}\right\}$

$\operatorname{Im} T = \left\{\begin{pmatrix}a&b\\b&-a\end{pmatrix}\right\}$ has basis $\left\{\begin{pmatrix}1&0\\0&-1\end{pmatrix}, \begin{pmatrix}0&1\\1&0\end{pmatrix}\right\}$

(d) T is not invertible as $\ker T \neq (0)$.

8\. (a) IV (b) V (c) I (d) VI (e) III

(9) Let A represent T with respect to the standard basis. Choose an orthonormal eigenbasis $\vec{v}_1, \dots, \vec{v}_n$ of $\mathbb{R}^n$ for A^TA. Say $A^TA\,\vec{v}_i = \lambda_i \vec{v}_i$. (Possible as A^TA is symmetric.)

Then $$T\vec{v}_i \cdot T\vec{v}_j = (A\vec{v}_i)^T(A\vec{v}_j) = \vec{v}_i^{\,T}(A^TA)\vec{v}_j$$
$$= \lambda_j\, \vec{v}_i \cdot \vec{v}_j$$
$$= \begin{cases} 0 & i \neq j \\ \lambda_j & i = j \end{cases}$$

Thus $\vec{v}_1, \dots, \vec{v}_n$ is an orthonormal basis and $T\vec{v}_1, \dots, T\vec{v}_n$ are orthogonal.