Mathematics 21b.
Linear Algebra and Differential Equations

Reflection Sheet

1. Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ denote an invertible linear transformation which operates in the following ways:

   - $T \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, and
   - $T \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

   (a) Find the matrix A which governs this transformation.

   (b) Find $T \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

   (c) Find $T^{-1} \begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

(d) **True or False?** In order to determine the matrix for a linear transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, it is enough to know $T(\vec{x})$ and $T(\vec{y})$ for any two vectors $\vec{x}, \vec{y} \in \mathbb{R}^2$. Justify your response.

2. Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ denote the linear transformation which is geometrically described as a reflection through the plane $x + y + z = 0$.

(a) Find all $\vec{x} \in \mathbb{R}^3$ such that $T(\vec{x}) = \vec{x}$.

(b) Find all $\vec{x} \in \mathbb{R}^3$ such that $T(\vec{x}) = -\vec{x}$.

(c) Find all $\vec{x} \in \mathbb{R}^3$ such that $T(\vec{x}) = \vec{0}$.

(d) Find all $\vec{x} \in \mathbb{R}^3$ such that $T(\vec{x}) = 3\vec{x}$.

(e) Is T invertible? Justify your answer.

3. Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ denote the linear transformation which is geometrically decsibed as a projection onto the plane $x + y + z = 0$.

   (a) Find all $\vec{x} \in \mathbb{R}^3$ such that $T(\vec{x}) = \vec{x}$.

   (b) Find all $\vec{x} \in \mathbb{R}^3$ such that $T(\vec{x}) = -\vec{x}$.

   (c) Find all $\vec{x} \in \mathbb{R}^3$ such that $T(\vec{x}) = \vec{0}$.

(d) Find all $\vec{x} \in \mathbb{R}^3$ such that $T(\vec{x}) = 3\vec{x}$.

(e) Is T invertible? Justify your answer.

4. Let $A \in \mathbb{R}^{2\times 2}$ denote the matrix which corresponds to a reflection through some line which passes through the origin. Arguing geometrically, determine what $A \times A$ is.

5. Let $A \in \mathbb{R}^2 \times \mathbb{R}^2$ denote the matrix which corresponds to a shear through some line which passes through the origin. Arguing geometrically, determine what $(A - I_2) \times (A - I_2)$ is. (This problem appeared on last year's first midterm.)

6. Find a 2×2 matrix B such that $B^{17} = I_2$, and $B^n \neq I_2$ when $1 \leq n \leq 16$, or explain why no such matrix exists.