

Name: _____

Math 21b Final Exam Tuesday, January 14th, 2002

Please circle your section:

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Question	Points	Score
1	20	
2	10	
3	6	
4	10	
5	12	
6	12	
7	6	
8	8	
9	10	
10	6	
Total	100	

You have three hours to take this final exam. Pace yourself by keeping track of how many problems you have left to go and how much time remains. You don't have to answer the problems in any particular order. So move on to another problem if you find you're stuck and that you are spending too much time on one problem.

To receive full credit on a problem, you will need to justify your answers carefully - unsubstantiated answers, even if correct, will receive little or no credit (except if the directions for that question specifically say no justification is necessary, such as the True/False).

Please be sure to write neatly - illegible answers will also receive little or no credit.

If more space is needed, use the back of the previous page to continue your work. Be sure to make a note of that so that the grader knows where to find your answers.

You are allowed one page of notes on it during the test, but you are not allowed to use any other references or calculators during this test.

Good luck! Focus and do well!

Question 1. (20 points total)

True or False (2 points each) No justification is necessary, simply circle **T** or **F** for each statement.

T **F** (a) If A is a 2×2 matrix with $\det(A) = 0$, then one column of A is a multiple of the other.

T **F** (b) The matrix $A^T A$ is symmetric for all matrices A .

T **F** (c) If A and B are invertible $n \times n$ matrices, then AB and BA are similar matrices.

T **F** (d) If A is an invertible matrix whose eigenvalues are all positive, then the eigenvalues of A^2 must be the same as the eigenvalues of A .

T **F** (e) If A is an 2×2 matrix with determinant 6, then the determinant of $2A$ must equal 12.

T F (f) If \vec{v}_1 and \vec{v}_2 are both eigenvectors for an $n \times n$ matrix A , then the sum, $\vec{v}_1 + \vec{v}_2$, must also be an eigenvector of A .

T F (g) If A is any (real) $n \times n$ matrix then $\det(A^T A)$ cannot be negative.

T F (h) There are invertible 3×3 matrices A and S such that $S^{-1}AS = -A$.

T F (i) If $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$ is an eigenbasis for both A and B , then A and B must be similar matrices.

T F (j) Suppose 0 is an eigenvalue for two $n \times n$ matrices, A and B . Suppose the geometric multiplicity of 0 is the same for both A and B , then A and B must also have the same rank.

Question 2 (10 points total)

Find the determinants for each of the following matrices (2 points each). Be sure to show all your work, and justify your answers (i.e. just writing down “0,” even if it’s correct, will not be considered a complete answer).

(a)
$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 3 & 3 & 3 \\ 1 & 1 & 2 & 0 \\ 2 & 0 & 2 & 2 \end{bmatrix}$$

(b)
$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 3 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{bmatrix}$$

(c) The 2×2 matrix representing a 210 degree rotation counterclockwise.

(d) The $n \times n$ matrix representing a dilation by a factor of 10.

(e) The 5×5 matrix representing a transformation $T(\vec{x})$ that has the effect of swapping the first two standard basis vectors, and that has no effect on the other standard basis vectors, i.e. $T(\vec{e}_1) = \vec{e}_2$, $T(\vec{e}_2) = \vec{e}_1$, and $T(\vec{e}_i) = \vec{e}_i$, for $i > 2$.

Question 3. (6 points total)

Find all values for k such that the following homogeneous linear system has nontrivial solutions (i.e. nonzero solutions).

$$\begin{aligned} x + 3y - 2z &= 0 \\ 2x + y + 3z &= 0 \\ 5x - 5y + kz &= 0 \end{aligned}$$

Question 4. (10 points total)

(a) (7 points) Find a basis for the orthogonal complement of the subspace of \mathbf{R}^4 spanned by vectors

$$\begin{bmatrix} \square \\ \square \\ \square \\ \square \\ \square \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 4 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} \square \\ \square \\ \square \\ \square \end{bmatrix} \begin{bmatrix} 4 \\ 6 \\ 2 \\ 3 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} \square \\ \square \\ \square \\ \square \\ \square \end{bmatrix} \begin{bmatrix} 3 \\ 7 \\ 6 \\ 4 \end{bmatrix}.$$

(b) (3 points) Suppose that A is a symmetric 6×6 matrix such that the image of A is equal to a 2-dimensional plane in \mathbf{R}^6 . Is it possible to determine whether or not 0 is an eigenvalue for A ? If so, is it also possible to determine both the algebraic and geometric multiplicities of 0? If it is possible, then find these, if it is not possible, then explain why not.

Question 5. (12 points total)

(a) (4 points) Let \mathbf{P}_n be the linear space of polynomials of degree n or less. Let $T : \mathbf{P}_3 \rightarrow \mathbf{P}_4$ be defined by $T(p(x)) = x^3 p'(x)$, where $p(x)$ is a polynomial in \mathbf{P}_3 . Show that T is a linear transformation.

(b) (2 points) Find a basis for the kernel of transformation T and determine its dimension.

Question 5 continued.

(c) (6 points) Consider the linear space V consisting of all 2×2 matrices for which the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector. Find a basis for this space and determine its dimension.

Question 6. (12 points total)

(a) (6 points) The matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ has eigenvectors $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Find an orthogonal matrix S , and a diagonal matrix D , so that $A = SDS^{-1}$

Question 6 continued.

(b) (6 points) Consider the matrix $B = \begin{bmatrix} 8 & 12 \\ 4 & 6 \end{bmatrix}$. Note that B has determinant equal to 0 and trace equal to 2. By diagonalizing B calculate B^{10} (note $2^{10} = 1,024$)

Question 7. (6 points total)

Suppose that $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k$ are eigenvectors of an $n \times n$ matrix A . Let V be the subspace spanned by $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k$. Show that if \vec{x} is a vector in V , then $A\vec{x}$ is in V as well.

(Note, $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k$ isn't necessarily an eigenbasis, as k might be less than n)

Question 8. (8 points total)

(a) (4 points) Given that $(x - 1)(x - 2)(x - 3) = x^3 - 6x^2 + 11x - 6$, find all solutions to the differential equation

$$2f''' - 12f'' + 22f' - 12f = 0$$

(b) (4 points) Find the solution to the differential equation in part (a) such that $f(0) = 2$, $f'(0) = 2$ and $f''(0) = 0$.

Question 9. (10 points total)

Let $A = \begin{bmatrix} 0 & 1 \\ b & c \end{bmatrix}$ where b and c are real numbers. Consider the continuous dynamical system

$$\frac{d\bar{x}}{dt} = A\bar{x}$$

(a) (4 points) What inequality involving b and c ensures that the solution to the system will have a phase portrait composed of trajectories spiraling inwards towards the origin?

(a) (6 points) Solve this continuous dynamical system if $b = 4$, $c = 5$, and $\bar{x}(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

(your answer should be a closed formula for $\bar{x}(t)$)

Question 10. (6 points total)

Find a symmetric 2 x 2 matrix, A , with the following properties:

(i) $\bar{x} = \begin{bmatrix} \square \\ \square \end{bmatrix}$ is an eigenvector for A

(ii) the sum of the two eigenvalues of A equals 0

(iii) the determinant of A equals -1.