Name: Answer Key

Math 21b Final Exam Tuesday, January 14th, 2002

Please circle your section:

Tom Judson	Andy Engelward	Andy Engelward
Katherine Visnjic (CA)	Jakub Topp (CA)	Erin Aylward (CA)
MWF 9-10	MWF 10-11	MWF 11-12

Question	Points	Score
1	20	
2	10	
3	6	
4	10	
5	12	
6	12	
7	6	
8	8	
9	10	
10	6	
Total	100	

You have three hours to take this final exam. Pace yourself by keeping track of how many problems you have left to go and how much time remains. You don't have to answer the problems in any particular order. So move on to another problem if you find you're stuck and that you are spending too much time on one problem.

To receive full credit on a problem, you will need to justify your answers carefully - unsubstantiated answers, even if correct, will receive little or no credit (except if the directions for that question specifically say no justification is necessary, such as the True/False).

Please be sure to write neatly - illegible answers will also receive little or no credit.

If more space is needed, use the back of the previous page to continue your work. Be sure to make a note of that so that the grader knows where to find your answers.

You are allowed one page of notes on it during the test, but you are not allowed to use any other references or calculators during this test.

Good luck! Focus and do well!

Question 1. (20 points total)

True or False (2 points each) No justification is necessary, simply circle **T** or **F** for each statement.

T **F** (a) If A is a 2 x 2 matrix with $\det(A) = 0$, then one column of A is a multiple of the other.

True – $\det = 0$ means $\operatorname{rank}(A) < 2$, columns are linearly dependent, or check $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\det = ad - bc$, so if $ad - bc = 0$ then $ad = bc$, and if $a \neq 0 \neq c$, then $\frac{b}{a} = \frac{d}{c}$, i.e. $\left(\frac{b}{a}\right)\begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix}$, if $a = 0$, then either b or $c = 0$, and the two columns are multiples of each other (by 0).

T **F** (b) The matrix A^TA is symmetric for all matrices A.

A matrix is symmetric if it equals its own transpose. So check $(A^TA)^T \stackrel{?}{=} A^TA$, yes, as $(A^TA)^T$
$= A^T(A^T)^T$ (reverse order of product)
$= A^TA$

T **F** (c) If A and B are invertible $n \times n$ matrices, then AB and BA are similar matrices.

Yes, we know M and N are similar if there exists an invertible matrix S with $SMS^{-1} = N$, here for AB and BA, try using $S = B$, then $\underbrace{B(AB)B^{-1} = BABB^{-1} = BA}_{\text{similar matrices}}$

T **F** (d) If A is an invertible matrix whose eigenvalues are all positive, then the eigenvalues of A^2 must be the same as the eigenvalues of A.

Not true, the eigenvalues of A^2 are simply the square of the eigenvalues of A, so for instance if $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ then A's eigenvalues are 1 and 2, but $A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$ with eigenvalues 1 and $(2)^2 = 4$

T **F** (e) If A is an 2 x 2 matrix with determinant 6, then the determinant of $2A$ must equal 12.

Be careful! multiplying one row of A by 2 increases the determinant by a factor of 2, so multiplying both rows (i.e. $2A$), increases the determinant by a factor of 4, so $\det(2A) = 24$

T **F** (f) If $\vec{v}_1$ and $\vec{v}_2$ are both eigenvectors for an $n \times n$ matrix A, then the sum, $\vec{v}_1 + \vec{v}_2$, must also be an eigenvector of A.

Not necessarily, if $\vec{v}_1$ and $\vec{v}_2$ are eigenvectors with the same eigenvalue, then yes, $\vec{v}_1+\vec{v}_2$ will also be an eigenvector, otherwise no. Consider $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ eigenvector, $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is an eigenvector but $\vec{v}_1+\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and $A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, so $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is not an eigenvector

T F (g) If A is any real $n \times n$ matrix then $\det(A^T A)$ cannot be negative.

$\det(A^T A) = \det(A^T)\det(A)$, but $\det(A^T) = \det(A)$

so $\det(A^T A) = (\det(A))^2$, which is ≥ 0

T **F** (h) There are invertible 3 x 3 matrices A and S such that $S^{-1}AS = -A$.

Not possible – check determinants,

$\det(S^{-1}AS) = \det(S^{-1})\det(A)\det(S) = \frac{1}{\det S} \det A \det S$

$= \det(A)$, but $\det(-A) = \det(-I_3)\det(A) = -\det(A)$

so $\det(A) = -\det(A)$, so $\det A = 0$, but then A can't be invertible

T **F** (i) If $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ is an eigenbasis for both A and B, then A and B must be similar matrices.

Not necessarily – if they had the same eigenvalues with same multiplicity, yes, but consider $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ each has eigenbasis $\vec{e}_1, \vec{e}_2$, but they are definitely not similar as $S^{-1}I_2S = I_2 \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

T F (j) Suppose 0 is an eigenvalue for two $n \times n$ matrices, A and B. Suppose the geometric multiplicity of 0 is the same for both A and B, then A and B must also have the same rank.

By rank-nullity, geo. multiplicity of $0 = \dim(\text{kernel})$,
$= \text{nullity}$, same for A, B

so since both A, B are $n \times n$, then

$\text{rank}(A) = \text{rank}(B) = n - \text{nullity}$

Question 2 (10 points total)

Find the determinants for each of the following matrices (2 points each). Be sure to show all your work, and justify your answers (i.e. just writing down "0," even if it's correct, will not be considered a complete answer).

(a) $\begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 3 & 3 & -3 \\ 1 & 1 & 2 & 0 \\ 2 & 0 & 2 & 2 \end{bmatrix}$

You can try finding the determinant by rref, but always check for linear independence first → clearly the 3rd and 4th columns are sum/difference of 1st two columns, so det = 0.

(b) $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & 3 & 0 \\ 1 & -1 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{bmatrix}$

This is almost a lower triangular matrix, if you swap the 1st and 4th rows, then swap the 2nd and 3rd rows you get $\begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & -1 & 3 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$ with determinant equal to -6.

Now since you made 2 row swaps, then the determinant changed by -1 twice, i.e. determinant is still -6 (or, could also see that only one pattern which is nonzero exists giving determinant -6)

(c) The 2 x 2 matrix representing a 210 degree rotation counterclockwise.

Actually check any rotation matrix: $\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

determinant $= \cos^2\alpha - (-\sin^2\alpha) = \cos^2\alpha + \sin^2\alpha = 1$

(d) The $n \times n$ matrix representing a dilation by a factor of 10.

So the matrix looks like $\underbrace{\begin{bmatrix} 10 & & & \\ & 10 & & \\ & & \ddots & \\ & & & 10 \end{bmatrix}}_{n \text{ cols}} \Big\} n \text{ rows}$ with a det of 10^n

(e) The 5 x 5 matrix representing a transformation $T(\vec{x})$ that has the effect of swapping the first two standard basis vectors, and that has no effect on the other standard basis vectors, i.e. $T(\vec{e}_1) = \vec{e}_2$, $T(\vec{e}_2) = \vec{e}_1$, and $T(\vec{e}_i) = \vec{e}_i$, for $i > 2$.

so T's matrix looks like $\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$. 1 row swap away from I_5, so determinant $= -1$

Question 3. (6 points total)

Find all values for k such that the following homogeneous linear system has nontrivial solutions (i.e. nonzero solutions).

$$\begin{Bmatrix} x & + & 3y & - & 2z & = & 0 \\ 2x & + & y & + & 3z & = & 0 \\ 5x & - & 5y & + & kz & = & 0 \end{Bmatrix}$$

For this system to have non-trivial solutions, then the rank of the coefficient matrix would have to be less than 3 (otherwise the augmented matrix in rref would just be $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$ with solution $x=y=z=0$). So calculate rref:

$$\begin{bmatrix} 1 & 3 & -2 \\ 2 & 1 & 3 \\ 5 & -5 & k \end{bmatrix} \begin{matrix} \\ -2(I) \\ -5(I) \end{matrix} \longrightarrow \begin{bmatrix} 1 & 3 & -2 \\ 0 & -5 & 7 \\ 0 & -20 & k+10 \end{bmatrix} \begin{matrix} \\ \div(-5) \\ -4(II) \end{matrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & -7/5 \\ 0 & 0 & k-18 \end{bmatrix}$$

We're not quite in rref yet, but it's clear that

This will have rank < 3 exactly when the last row $[0 \ 0 \ k-18]$ is $[0 \ 0 \ 0]$, or when $k = 18$

This is the only value for k that leads to rank < 3, otherwise if $k-18 \neq 0$, then $\text{rref} = I_3$, (and the solution is just the trivial $x=y=z=0$).

Question 4. (10 points total)

(a) (7 points) Find a basis for the orthogonal complement of the subspace of $\mathbf{R}^4$ spanned by vectors

$$\begin{bmatrix} 1 \\ -3 \\ -4 \\ 3 \end{bmatrix}, \begin{bmatrix} -4 \\ 6 \\ -2 \\ 3 \end{bmatrix}, \text{ and } \begin{bmatrix} -3 \\ 7 \\ 6 \\ -4 \end{bmatrix}.$$

So if $A = \begin{bmatrix} 1 & -4 & -3 \\ -3 & 6 & 7 \\ -4 & -2 & 6 \\ 3 & 3 & -4 \end{bmatrix}$ then we're looking for $(\mathrm{Im}\,A)^\perp$, which is just equal to $\ker(A^T)$

$\ker(A^T)$: $\begin{bmatrix} 1 & -3 & -4 & 3 \\ -4 & 6 & -2 & 3 \\ -3 & 7 & 6 & -4 \end{bmatrix} \begin{matrix} \\ +4(I) \\ +3(I) \end{matrix} \longrightarrow \begin{bmatrix} 1 & -3 & -4 & 3 \\ 0 & -6 & -18 & 15 \\ 0 & -2 & -6 & 5 \end{bmatrix} \begin{matrix} \\ \div(-6) \\ -\frac{1}{3}(II) \end{matrix}$

$\hookrightarrow \begin{bmatrix} 1 & -3 & -4 & 3 \\ 0 & 1 & 3 & -5/2 \\ 0 & 0 & 0 & 0 \end{bmatrix} + 3(II) \longrightarrow \begin{bmatrix} 1 & 0 & 5 & -9/2 \\ 0 & 1 & 3 & -5/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$x_1 = -5s + 9/2\,t$
$x_2 = -3s + 5/2\,t$
$x_3 = s, \quad x_4 = t$

so $\ker(A^T) = s\begin{bmatrix} -5 \\ -3 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 9/2 \\ 5/2 \\ 0 \\ 1 \end{bmatrix}$, so a basis for the orthogonal complement is given by

$$\begin{bmatrix} -5 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 9 \\ 5 \\ 0 \\ 2 \end{bmatrix}$$

(b) (3 points) Suppose that A is a symmetric 6 x 6 matrix such that the image of A is equal to a 2-dimensional plane in $\mathbf{R}^6$. Is it possible to determine whether or not 0 is an eigenvalue for A? If so, is it also possible to determine both the algebraic and geometric multiplicities of 0? If it is possible, then find these, if it is not possible, then explain why not.

so $\dim(\mathrm{Image}(A)) = 2 = \mathrm{rank}(A)$, so nullity + rank $= 6$
$\Rightarrow$ nullity $= \dim(\ker(A)) = 4$,

Since A is symmetric, it has an eigenbasis, and all algebraic and geometric multiplicities are equal, so geo. multiplicity of the 0 eigenvalue $= \dim(\ker(A)) = 4 =$ algebraic multiplicity of 0, (and yes 0 is an eigenvalue for A)

Question 5. (12 points total)

(a) (4 points) Let $\mathbf{P}_n$ be the linear space of polynomials of degree n or less. Let $T : \mathbf{P}_3 \to \mathbf{P}_4$ be defined by $T(p(x)) = x^3 p''(x)$, where $p(x)$ is a polynomial in $\mathbf{P}_3$. Show that T is a linear transformation.

Check: is $T(p(x)+q(x)) = T(p(x)) + T(q(x))$ for $p, q \in P_3$?

$$T(p(x)+q(x)) = x^3 (p(x)+q(x))'' = x^3 (p''(x) + q''(x))$$
$$= x^3 p''(x) + x^3 q''(x) = T(p(x)) + T(q(x))$$

Next is $T(k\,p(x)) = k\,T(p(x))$ for all $k \in \mathbb{R}$, $p(x) \in P_3$?

well $T(k(p(x))) = x^3 (k\,p(x))'' = x^3 k\,p''(x) = k(x^3 p''(x))$
$$= k\,T(p(x)) \checkmark$$

So yes $T(p(x)) = x^3 p''(x)$ is a linear transformation

(b) (2 points) Find a basis for the kernel of transformation T and determine its dimension.

What $p(x)$ is sent to 0 by T?

$T(p(x)) = 0$ if and only if $p''(x) = 0$, i.e.

only if $p(x) = ax + b$ (for higher degree polynomials, $p''(x) \neq 0$, so $T(p(x)) \neq 0$)

Thus a basis for the kernel of T is given by $\{1, x\}$, and dimension is 2

Question 5 continued.

(c) (6 points) Consider the linear space V consisting of all 2 x 2 matrices for which the vector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigenvector. Find a basis for this space and determine its dimension.

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be in V, then

$A \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} k \\ -k \end{bmatrix}$ for some value k,

so $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} a-b \\ c-d \end{bmatrix} = \begin{bmatrix} k \\ -k \end{bmatrix}$,

thus $a-b=k$, $c-d=-k$, or so $a-b=-(c-d)$
$= d-c$

so $a = b+d-c$. Other than this condition, though, any 2×2 matrix with $a=b+d-c$, will work,

so matrix A is of the form $\begin{bmatrix} b+d-c & b \\ c & d \end{bmatrix}$ $b,c,d \in \mathbb{R}$

or splitting this up as a linear combination:

$b\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$,

so a basis for V is $\left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

and $\dim(V) = 3$

Question 6. (12 points total)

(a) (6 points) The matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ has eigenvectors $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Find an orthogonal matrix S, and a diagonal matrix D, so that $A = SDS^{1}$

check the eigenvalues of the 3 eigenvectors:

$A\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ with e.value $= 1$, $\quad A\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, e.value $= 1$ again

and $A\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}$ e.value $= 4$. So to find an orthogonal matrix S, we can't just take these three vectors, we need to orthonormalize (i.e. Gram-Schmidt) first. Note, however that the 3rd e.vector is already $\perp$ to the first two (e.vectors with different e.values for a symmetric matrix are always perpendicular to each other). So we need to G-S the first two vectors:

$\vec{u}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$, to find $\vec{u}_2$, take $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - \left(\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}\right)\begin{bmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$

$$= \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1/2 \\ -1/2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1/2 \\ -1/2 \end{bmatrix}$$

and so $\vec{u}_2 = \begin{bmatrix} 2/\sqrt{6} \\ -1/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix}$. Normalizing $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ yields $\begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$

so S can be given by $\begin{bmatrix} 0 & 2/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}$

then $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

Question 6 continued.

(b) (6 points) Consider the matrix $B = \begin{bmatrix} 8 & 12 \\ -4 & -6 \end{bmatrix}$. Note that B has determinant equal to 0 and trace equal to 2. By diagonalizing B calculate B^{10} (note $2^{10} = 1{,}024$)

So B has eigenvalues 0 and 2 (from $\det = 0$, Trace $= 2$)

eigenvectors : $\lambda I_2 - B$: $\begin{bmatrix} -8 & -12 \\ 4 & 6 \end{bmatrix} \longrightarrow$ kernel = span $\begin{bmatrix} 3 \\ -2 \end{bmatrix} = E_0$

$E_2 = \ker(2I_2 - B) = \ker \begin{bmatrix} -6 & -12 \\ 4 & 8 \end{bmatrix} = \text{span} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

So take $S = \begin{bmatrix} 3 & 2 \\ -2 & -1 \end{bmatrix}$ then $S^{-1} = \begin{bmatrix} -1 & -2 \\ 2 & 3 \end{bmatrix}$, and $D = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$

so $B = SDS^{-1}$, and $B^{10} = SD^{10}S^{-1}$

$$= \begin{bmatrix} 3 & 2 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 2^{10} \end{bmatrix} \begin{bmatrix} -1 & -2 \\ 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 2(1024) \\ 0 & -1024 \end{bmatrix} \begin{bmatrix} -1 & -2 \\ 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 4{,}096 & 6{,}144 \\ -2048 & -3072 \end{bmatrix}$$

Question 7. (6 points total)

Suppose that $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ are eigenvectors of an $n \times n$ matrix A. Let V be the subspace spanned by $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$. Show that if $\vec{x}$ is a vector in V, then $A\vec{x}$ is in V as well.
(Note, $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ isn't necessarily an eigenbasis, as k might be less than n)

If $\vec{x}$ is in V, then since V is spanned by $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$, then $\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k$, for some $c_1, c_2, \ldots, c_k$, then $A\vec{x} = A(c_1\vec{v}_1 + \cdots + c_k\vec{v}_k)$

$$= c_1 A\vec{v}_1 + \cdots + c_k A\vec{v}_k,$$

now since the $\vec{v}_i$ are all eigenvectors for A, (suppose with respective eigenvalues λ_i),

then $A\vec{x} = \ldots = c_1(A\vec{v}_1) + \cdots + c_k(A\vec{v}_k)$

$$= (c_1\lambda_1)\vec{v}_1 + \cdots + (c_k\lambda_k)\vec{v}_k$$

~~so~~ $A\vec{x}$ is also a linear combination of the $\vec{v}_i$,

so $A\vec{x}$ is also in V

Question 8. (8 points total)

(a) (4 points) Given that $(x-1)(x-2)(x-3) = x^3 - 6x^2 + 11x - 6$, find all solutions to the differential equation

$$2f''' - 12f'' + 22f' = 12f$$

So, solve $2f''' - 12f'' + 22f' - 12f = 0$,

or $f''' - 6f'' + 11f' - 6f = 0$,

characteristic polynomial of this linear differential equation is $x^3 - 6x^2 + 11x - 6$, with roots $1, 2$, and 3 (given by the factorization), so the general solution is given by $f(t) = c_1 e^{t} + c_2 e^{2t} + c_3 e^{3t}$

for c_1, c_2, c_3 constants

(b) (4 points) Find the solution to the differential equation in part (a) such that $f(0) = 2$, $f'(0) = 2$ and $f''(0) = 0$.

Using $f(t) = c_1 e^{t} + c_2 e^{2t} + c_3 e^{3t}$

then $f'(t) = c_1 e^{t} + 2c_2 e^{2t} + 3c_3 e^{3t}$

$f''(t) = c_1 e^{t} + 4c_2 e^{2t} + 9c_3 e^{3t}$

and $f(0) = c_1 + c_2 + c_3 = 2$

$f'(0) = c_1 + 2c_2 + 3c_3 = 2$

$f''(0) = c_1 + 4c_2 + 9c_3 = 0$

So solve $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & 2 & 3 & 2 \\ 1 & 4 & 9 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 3 & 8 & -2 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & -2 \end{array}\right]$

$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right] \Rightarrow c_1 = 1, \; c_2 = 2, \; c_3 = -1,$

so the solution is $f(t) = e^{t} + 2e^{2t} - e^{3t}$

Question 9. (10 points total)

Let $A = \begin{bmatrix} 0 & 1 \\ -b & -c \end{bmatrix}$ where b and c are real numbers. Consider the continuous dynamical system

$$\frac{d\bar{x}}{dt} = A\bar{x}$$

(a) (4 points) What inequality involving b and c ensures that the solution to the system will have a phase portrait composed of trajectories spiraling inwards towards the origin?

characteristic polynomial $= \det(\lambda I_2 - A) = \lambda(\lambda + c) + b$
$= \lambda^2 + c\lambda + b$

eigenvalues are $\dfrac{-c \pm \sqrt{c^2 - 4b}}{2}$

Phase portrait looks like  (inward spiraling towards origin)

when the eigenvalues are complex, with negative real part,
so $c^2 - 4b < 0$, and $c > 0$ gives this result.

(a) (6 points) Solve this continuous dynamical system if $b = 4$, $c = 5$, and $\bar{x}(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

(your answer should be a closed formula for $\bar{x}(t)$)

So we need to find eigenvalues: $\dfrac{-c \pm \sqrt{c^2 - 4b}}{2} = \dfrac{-5 \pm \sqrt{25 - 16}}{2}$

$= -4$ and -1

Eigenvectors: $\ker(-4I_2 - A) = \ker \begin{bmatrix} -4 & -1 \\ +4 & +1 \end{bmatrix} = \text{span} \begin{bmatrix} 1 \\ -4 \end{bmatrix}$

$\ker(-1I_2 - A) = \ker \begin{bmatrix} -1 & -1 \\ 4 & 4 \end{bmatrix} = \text{span} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

so $\vec{x}(t) = c_1 e^{-4t} \begin{bmatrix} 1 \\ -4 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\vec{x}(0) = c_1 \begin{bmatrix} 1 \\ -4 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\hookrightarrow = \begin{bmatrix} 1 & 1 \\ -4 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$, so $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -1/3 & -1/3 \\ 4/3 & 1/3 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

so $\vec{x}(t) = -e^{-4t} \begin{bmatrix} 1 \\ -4 \end{bmatrix} + 3e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3e^{-t} - e^{-4t} \\ -3e^{-t} + 4e^{-4t} \end{bmatrix}$

Question 10. (6 points total)

Find a symmetric 2 x 2 matrix, A, with the following properties:

(i) $\vec{x} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is an eigenvector for A

(ii) the sum of the two eigenvalues of A equals 0

(iii) the determinant of A equals -1.

So since A is symmetric, $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$ is also an eigenvector (perpendicular to $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$)

then sum of eigenvalues is 0 implies eigenvalues are $k, -k$, and $\det(A) = -1$ implies $k(-k) = -k^2 = -1$, so k is 1 or -1,

Then one possibility is $A\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $A\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

e. value = 1 e. value = -1

so $A\begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & -3 \end{bmatrix}$

so $A = \frac{1}{10}\begin{bmatrix} 3 & 1 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix}$ since $\frac{1}{10}\begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix}$ is the inverse of $\begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}$

$= \frac{1}{10}\begin{bmatrix} 8 & 6 \\ 6 & -8 \end{bmatrix}$

$= \begin{bmatrix} .8 & .6 \\ .6 & -.8 \end{bmatrix}$