MATH 21b **FALL 2001**

FINAL EXAMINATION

Wednesday, 23 January, 2002.

Name: ______________________________

Teaching Fellow (PLEASE CIRCLE)

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Instructions:

1. Do not open this test until told to do so.
2. Please do not detach any pages from this exam.
3. You may use your calculator and one (1) page of notes not exceeding 8.5 by 11 inches in size.
4. You may use the backs of test sheets for scratch paper, or to continue your working on problems. If you write on the backs of the test sheets, **please label your working very clearly**.
5. SHOW ALL YOUR WORK.
6. Exam proctors are not permitted to answer questions regarding the content of the test.
7. Many of the questions have precisely worded instructions. **Be sure to read all instructions carefully,** and do all that is asked.
8. May the Force be with you.

Problem	Total	Score
1	10	
2	11	
3	23	
4	15	
5	19	
6	15	
7	16	
8	18	
9	12	
10	11	
Total	150	

1. (10 points total)

In this problem, $T:\mathbf{R}^4 \to \mathbf{R}^4$ is a linear transformation defined by the matrix equation:

$$T(\vec{x}) = A\vec{x}$$

where A is a 4 by 4 matrix. The Reduced Row Echelon Form (RREF) of A is given below.

$$RREF(A) = \begin{bmatrix} 1 & 2 & 0 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) **(4 points)** If possible, find a basis for the kernel of $T:\mathbf{R}^4 \to \mathbf{R}^4$. If you do not believe that this is possible, explain why.

(b) **(4 points)** If possible, find a basis for the image of $T:\mathbf{R}^4 \to \mathbf{R}^4$. If you do not believe that this is possible, explain why.

(c) **(2 points)** If possible, find the dimensions of the kernel and image of $T:\mathbf{R}^4 \to \mathbf{R}^4$. If you do not believe that this is possible, explain why.

2. (11 points total)

Recall that P_2 is the vector space (or linear space) consisting of all polynomials with degree less than or equal to 2.

(a) **(2 points)** Suppose that V is a vector space (or linear space) and that $\{\vec{v}_1,\ldots,\vec{v}_p\}$ is a set of vectors from V. What properties must the set $\{\vec{v}_1,\ldots,\vec{v}_p\}$ have in order to be a basis for V?

(b) **(6 points)** Show that the set of vectors:

$$\{\, f_1(x) = 1 + x,\ f_2(x) = 1 + x^2,\ f_3(x) = x + x^2 \,\}$$

is a basis for P_2.

(c) **(3 points)** Express: $f(x) = 2 - x + x^2$ as a linear combination of the vectors $f_1(x)$, $f_2(x)$ and $f_3(x)$.

3. (23 points total)

In this problem the matrix A will always refer to the 4 by 4 matrix given below.

$$A = \begin{bmatrix} 1 & 3 & 7 & 11 \\ 0 & \frac{1}{2} & 3 & 8 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

(a) (8 points) Complete the table given below. (Note that we have left more spaces than you need so don't be worried if some spaces are left blank.)

Eigenvalue of A	Algebraic Multiplicity	Geometric Multiplicity

(b) (3 points) Is the matrix A diagonalizable? Briefly indicate how you know.

Continued on the next page.

(c) **(6 points)** Complete the table given below.

Vector	Is this an eigenvector of A?	Corresponding eigenvalue
$\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$		
$\begin{bmatrix}-6\\1\\0\\0\end{bmatrix}$		
$\begin{bmatrix}7\\0\\0\\1\end{bmatrix}$		
$\begin{bmatrix}11\\-6\\1\\0\end{bmatrix}$		
$\begin{bmatrix}12\\0\\2\\0\end{bmatrix}$		
$\begin{bmatrix}159\\28\\6\\3\end{bmatrix}$		

(d) **(6 points)** Find a 4 by 4 matrix P that satisfies the equation:

$$P^{-1}AP = \begin{bmatrix}2 & 0 & 0 & 0\\0 & \frac{1}{2} & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 1\end{bmatrix}.$$

4. (15 points total)

In this problem $x(t)$ and $y(t)$ are differentiable functions that are defined by the equations:

$$2 \cdot \frac{dx}{dt} + 5 \cdot \frac{dy}{dt} = t$$

$$\frac{dx}{dt} + 3 \cdot \frac{dy}{dt} = 7 \cdot \cos(t).$$

The following integration formulas may be of some use in this problem:

(I) $\int t\,dt = \frac{1}{2}t^2 + C$ (II) $\int \cos(t)dt = \sin(t) + C.$

(a) **(4 points)** Re-write the system of differential equations as an equation involving a matrix and the vectors:

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} \text{ and } \begin{bmatrix} t \\ 7 \cdot \cos(t) \end{bmatrix}.$$

(b) **(3 points)** Is the matrix from Part (a) invertible? If so, find the inverse matrix. If not, provide evidence to show that the matrix is not invertible.

Continued on the next page.

(c) **(4 points)** Find equations for $\frac{dx}{dt}$ and $\frac{dy}{dt}$ that involve only constants, t and $\cos(t)$.

(d) **(4 points)** Suppose that you are given the initial conditions:

$$x(0) = 0 \qquad \text{and} \qquad y(0) = 0.$$

Find explicit formulas for the functions $x(t)$ and $y(t)$.

5. (19 points total)

In this problem, $f(x)$ will always refer to:

$$f(x) = e^x.$$

Consider the portion of this function that lies between $x = -\pi$ and $x = \pi$. The Fourier series of this portion of $f(x)$ has the form:

$$a_0 + \sum_{n=1}^{\infty} a_n \cdot \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \cdot \sin\left(\frac{n\pi x}{L}\right).$$

(a) **(5 points)** Find the value of the constant a_0 in the Fourier series.

(b) **(5 points)** Find the value of the constant a_1 in the Fourier series.

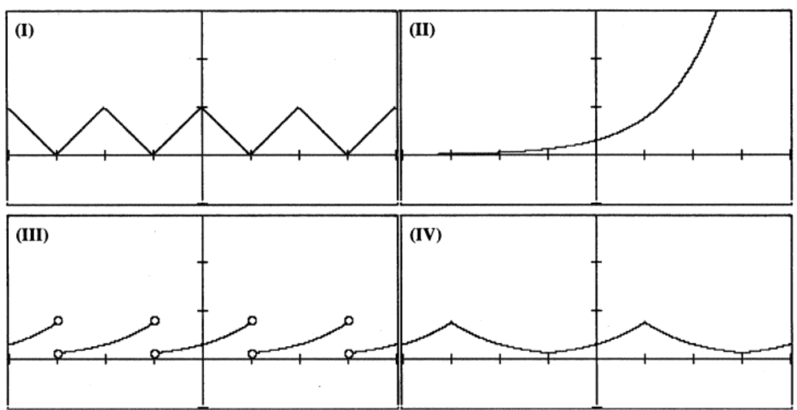
NOTE: $$\int e^x \cdot \cos(a \cdot x)dx = \frac{e^x \cdot \cos(a \cdot x)}{1+a^2} + \frac{a \cdot e^x \cdot \sin(a \cdot x)}{1+a^2} + C.$$

Continued on the next page.

(c) **(5 points)** Find the value of the constant b_1 in the Fourier series.

NOTE: $$\int e^x \cdot \sin(a \cdot x)dx = \frac{e^x \cdot \sin(a \cdot x)}{1+a^2} - \frac{a \cdot e^x \cdot \cos(a \cdot x)}{1+a^2} + C.$$

(d) **(4 points)** One of the graphs given below is the graph of $y = f(x)$ and one of the graphs is the graph of the Fourier series. Determine which graph is the graph of $y = f(x)$ and which graph is the graph of the Fourier series.



GRAPH OF $y = f(x)$: I II III IV (circle one)

GRAPH OF FOURIER SERIES: I II III IV (circle one)

6. (15 points total)

For each of the following statements decide whether the statement is true or false. If you believe that the statement is true, justify the statement. If you believe that the statement is false, give an example to show that it is false.

(a) **(3 points)** Suppose that A is an n by n matrix and that $A^2 = 0$. Then A is not invertible.

(b) **(3 points)** Suppose that A is an invertible n by n matrix. Then: $\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$.

(c) **(3 points)** Suppose that A and B are n by n matrices and that A is an invertible matrix. Suppose further that A and B commute – that is, $AB = BA$. Then A^{-1} and B commute.

(d) **(3 points)** Let A be an m by n matrix and let $\vec{b}$ be a non-zero vector in $\mathbf{R}^m$. The set of vectors, $\vec{x}$, that solve the equation:

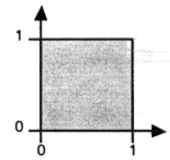
$$A\vec{x} = \vec{b}$$

is a subspace of $\mathbf{R}^n$.

(e) **(3 points)** Let A be an invertible n by n matrix. Then: $\det(A^T A^{-1}) \geq 0$.

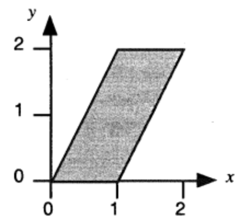
7. (16 points total)

Each of the diagrams shown below is obtained from the unit square:

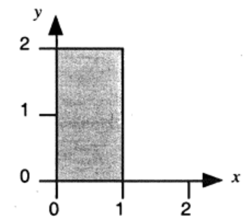


by some kind of transformation $T: \mathbf{R}^2 \rightarrow \mathbf{R}^2$. For each of the diagrams shown below, decide whether or not the transformation was a linear transformation or not. If you believe that the transformation was a linear transformation, find a matrix that could represent that transformation. If you believe that the transformation was not a linear transformation, explain why not.

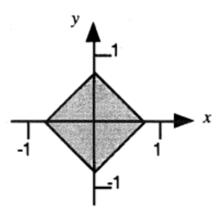
(a) (4 points)



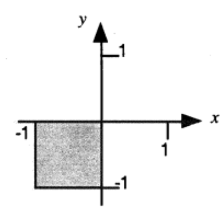
(b) (4 points)



(c) (4 points)



(d) (4 points)



8. (18 points total)

The point of this problem is to find the formula for a non-trivial function $u(x, t)$ that satisfies all of the following conditions:

- $\frac{\partial u}{\partial t} = 3 \cdot \frac{\partial^2 u}{\partial x^2}$ when $0 \le x \le \pi$ and $t \ge 0$.
- $u(0, t) = u(\pi, t) = 0$ when $t > 0$.
- $u(x, 0) = \sin(10 \cdot x)$ when $0 \le x \le \pi$.

(a) **(5 points)** Assuming a solution of the form: $u(x, t) = X(x) \cdot T(t)$, find explicit formulas for the functions $X(x)$ and $T(t)$ so that:

$$\frac{\partial u}{\partial t} = 3 \cdot \frac{\partial^2 u}{\partial x^2}.$$

NOTE: Your formulas may contain constants whose values are not specified.

(b) **(4 points)** Apply the boundary conditions: $u(0, t) = u(\pi, t) = 0$ to simplify the equations for $X(x)$ and $T(t)$ as much as possible.

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(c) **(3 points)** Calculate the Fourier series of: $f(x) = \sin(10 \cdot x)$. If you use any insights or mathematical theorems to expedite your calculations, briefly note them as part of your answer.

(d) **(6 points)** Combine your answers from Parts (a), (b) and (c) to write down a formula for the non-trivial function $u(x, t)$ that satisfies all of the conditions given at the beginning of this problem.

9. (12 points total)

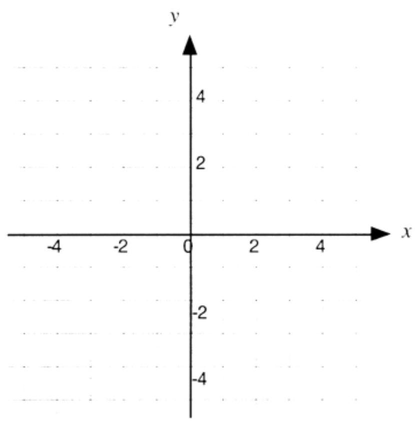
In this problem you may assume the information given in the table below. (There is no need for you to calculate the results given in the table.)

Matrix	$A = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}$	$B = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}$
Eigenvalues	$1, -1$	$2+i, 2-i$
Corresponding Eigenvectors	$\begin{bmatrix} -3 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1+i \\ -2 \end{bmatrix}, \begin{bmatrix} 1-i \\ -2 \end{bmatrix}$

(a) **(4 points)** Find an explicit solution to the discrete dynamical system:

$$\begin{bmatrix} x(t+1) \\ y(t+1) \end{bmatrix} = A \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

(b) **(2 points)** Using the axes provided, sketch the trajectory of the solution from Part (a).



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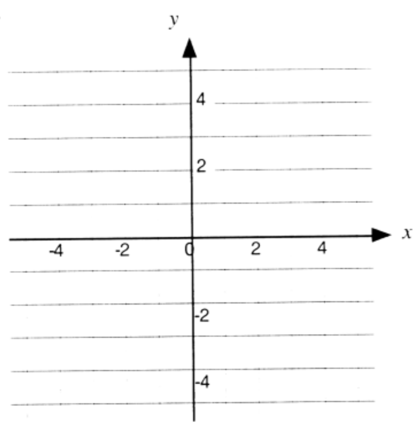
(c) **(4 points)** Find an explicit solution to the discrete dynamical system:

$$\begin{bmatrix} x(t+1) \\ y(t+1) \end{bmatrix} = B\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

(d) **(2 points)** Using the axes provided, sketch the trajectory of the solution from Part (c).



10. (11 points total)

Let A be an n by n matrix:

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

with the property that the entries in each row of the matrix add up to zero. That is:

$$\begin{aligned} a_{11} + a_{12} + \ldots + a_{1n} &= 0 \\ a_{21} + a_{22} + \ldots + a_{2n} &= 0 \\ \vdots \quad\quad \vdots \quad\quad\quad\quad \vdots \\ a_{n1} + a_{n2} + \ldots + a_{nn} &= 0. \end{aligned}$$

Prove that the determinant of the n by n matrix A must be equal to zero.

NOTE: For full credit, your proof must work for any n by n matrix with the given property. Limited partial credit may be available for proofs that work in the 2 by 2 or 3 by 3 cases.