

Name: _____

Math 21b Final Exam—Thursday, May 15, 2003

Please circle your section:

Thomas Judson	Thomas Judson	Kalle Karu
Eduardo Saverin (CA)	Rene Shen (CA)	Mark Bandstra (CA)
Jakob Topp (CA)	Albert Wang (CA)	MWF 12-1
MWF 10-11	MWF 11-12	

Ken Chung	Spiro Karigiannis
Nathan Lange (CA)	Jeff Berton (CA)
TuTh 10-11:30	TuTh 11:30-1

Problem Number	Possible Points	Score
1	20	
2	7	
3	9	
4	9	
5	8	
6	11	
7	10	
8	8	
9	8	
10 or 11	10	
Total	100	

Directions—Please Read Carefully! You have three hours to take this exam. Pace yourself by keeping track of how many problems you have left to go and how much time remains. You do not have to answer the problems in any particular order, so move to another problem if you find you are stuck or spending too much time on a single problem. To receive full credit on a problem, you will need to justify your answers carefully—unsubstantiated answers will receive little or no credit unless the directions for the question specifically say no justification is necessary, such as in the True/False section. Be sure to write neatly—illegible answers will receive little or no credit. If more space is needed, use the back of the previous page to continue your work. Be sure to make a note of this on the problem page so that the grader knows where to find your answers. You are allowed one 8-1/2 by 11 sheet of notes. No calculators or other aids are allowed. **Good Luck on the Exam!**

1. (20 points) True or False. No justification is necessary, simply circle **T** or **F** for each statement.

T **F** (a) If W is a subspace of \mathbb{R}^n and \mathbf{x} is not in W , then $\mathbf{x} - \text{proj}_W \mathbf{x}$ is not zero.

T **F** (b) A linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is completely determined by what T does to each column of the $n \times n$ identity matrix.

T **F** (c) If the matrix A^2 is diagonalizable, then A is diagonalizable.

T **F** (d) Similar matrices have the same eigenvectors.

T **F** (e) If A is similar to B , then A^2 is similar to B^2 .

T F (f) Let A be a 4×4 matrix with three distinct eigenvalues. If one of the eigenspaces is two dimensional, then A is diagonalizable.

T F (g) There exists a 5×5 matrix with no real eigenvalues.

T F (h) An orthogonal matrix is orthogonally diagonalizable.

T F (i) If W_1 and W_2 are subspaces of a linear space V , then $W_1 \cup W_2$ is always a subspace of V . (The set $W_1 \cup W_2$ is the set of all elements in V that are in either W_1 or W_2 or in both W_1 and W_2 .)

T F (j) If a real matrix A has only the eigenvalues 1 and -1 , then A is an orthogonal matrix.

2. Let $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ be an orthonormal basis for \mathbb{R}^n , and let $\lambda_1, \dots, \lambda_n$ be any real scalars. Define A to be the $n \times n$ matrix

$$A = \lambda_1 \mathbf{u}_1 \mathbf{u}_1^T + \dots + \lambda_n \mathbf{u}_n \mathbf{u}_n^T.$$

- (a) Show that A is symmetric. (2 points)

- (b) Show that $\lambda_1, \dots, \lambda_n$ are eigenvalues of A . (5 points)

3. Let

$$A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}.$$

- (a) Find an eigenbasis for A . [*Hint*: One of the eigenvalues of A is 7.] (4 points)

(b) Orthogonally diagonalize A (5 points)

4. (a) Let A be an $m \times n$ matrix such that $A^T A$ is invertible. Show that the columns of A are linearly independent. Be careful, A may not be a square matrix. (5 points)

(b) Show that the matrix

$$B = \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix}$$

is invertible if and only if a , b , and c are distinct real numbers. *You do not need to use part (a) to solve this problem.* (4 points)

5. We wish to find the function

$$y = c_1x + c_2 \cos\left(\frac{\pi x}{2}\right)$$

that best fits the following data using least-squares.

$$\begin{array}{c|cccc} x & 0 & 1 & 2 & 3 \\ \hline y & 0 & 2 & 2 & 4 \end{array}$$

(a) Write the matrix equation that must be solved. (4 points)

(b) Find the least-squares solution and write the corresponding function $f(x)$. (4 points)

6. Consider the dynamical system

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x},$$

where

$$A = \begin{bmatrix} -3 & 2 \\ -1 & -1 \end{bmatrix}$$

(a) Find the solution of $d\mathbf{x}/dt = A\mathbf{x}$ with the initial condition

$$\mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Be sure to state your solution in terms of real numbers and real functions and not the complex exponential function. (5 points)

(b) Sketch the graph of the trajectory for a solution of $dx/dt = Ax$. (2 points)

- (c) In the redwood forests of northern California dusky-footed wood rats provide up to 80% of the diet of the spotted owl, the main predator of the wood rat. Denote the owl and wood rat populations at time k by $\mathbf{x}_k = [O_k, R_k]^T$, where O_k is the number of owls in the region and R_k is the number of rats (measured in thousands). Suppose that the owl and rat population evolve according to the discrete dynamical system

$$\begin{aligned}O_{k+1} &= 0.5O_k + 0.4R_k \\R_{k+1} &= -pO_k + 1.1R_k,\end{aligned}$$

where $p > 0$ is a parameter to be specified. If $p = 0.104$, the eigenvalues of the matrix for the matrix of the dynamical system are $\lambda_1 = 1.02$ and $\lambda_2 = 0.58$. The eigenvectors for λ_1 and λ_2 are

$$\mathbf{v}_1 = \begin{bmatrix} 10 \\ 13 \end{bmatrix} \text{ and } \mathbf{v}_2 = \begin{bmatrix} 5 \\ 1 \end{bmatrix},$$

respectively. What is the long-term monthly growth (or decay) rate of the owl and wood rat populations. Be specific and carefully justify your conclusion. (4 points)

7. (a) Let $\mathbf{u} = \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix}$ be a unit vector in \mathbb{R}^2 . Give a 2×2 matrix R_θ which represents reflection along the line spanned by \mathbf{u} . (4 points)

(b) Let

$$\mathbf{u}_1 = \begin{bmatrix} \cos \theta_1 \\ \sin \theta_1 \end{bmatrix} \text{ and } \mathbf{u}_2 = \begin{bmatrix} \cos \theta_2 \\ \sin \theta_2 \end{bmatrix}$$

be two unit vectors in \mathbb{R}^2 . Show that the composition matrix $R_{\theta_2}R_{\theta_1}$ is a counterclockwise rotation by some angle. What is this angle in terms of θ_1 and θ_2 ? (4 points)

- (c) Now consider an arbitrary orthogonal $n \times n$ matrix A . Show that $\det(A) = \pm 1$. If $\det(A) = +1$, we call A a rotation. If $\det(A) = -1$, we call A a reflection. Show that the product of any two reflections is a rotation. *This part does not require any complicated calculations.* (2 points)

8. The *trace* of a $n \times n$ matrix is the sum of its diagonal elements.

- (a) Show that the trace of FG is equal to the trace of GF for two $n \times n$ matrices F and G .
(4 points)

- (b) If A and B are similar matrices, show that the trace of A is equal to the trace of B .
(4 points)

9. Let U be the subspace of \mathbf{R}^4 defined by the equations

$$\begin{aligned}2x_1 + x_2 + x_3 + x_4 &= 0 \\x_1 - x_2 + x_3 - x_4 &= 0,\end{aligned}$$

and let V be the subspace of \mathbf{R}^4 defined by the equations

$$\begin{aligned}-x_1 + x_2 + 2x_3 - 2x_4 &= 0 \\x_1 + 2x_2 + 2x_4 &= 0.\end{aligned}$$

- (a) Find a basis for $U \cap V$, where $U \cap V$ is the set of vectors in \mathbb{R}^4 that are in both U and V . (4 points)

(b) Find a basis for $(U \cap V)^\perp$. (4 points)

10. You only need to do Problem 10 or Problem 11. Credit will be given for only one of these two problems. Circle Problem 10 on the first page if you wish this problem to be graded.

(a) Solve the equation $x'' - 3x' + 2x = 0$, if $x(0) = 1$ and $x'(0) = -1$. (4 points)

(b) Consider the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}.$$

Show that

$$u(x, t) = b_n e^{-n^2 t} \sin nx$$

is a solution satisfying the boundary conditions $u(0, t) = u(\pi, t) = 0$ for $n = 1, 2, 3, \dots$, where b_n is an arbitrary constant. (3 points)

(c) Using the initial condition,

$$u(x, 0) = f(x) = 80 \sin^3 x = 60 \sin x - 20 \sin 3x.$$

solve the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

with boundary conditions $u(0, t) = u(\pi, t) = 0$. (3 points)

11. You only need to do Problem 10 or Problem 11. Credit will be given for only one of these two problems. Circle Problem 11 on the first page if you wish this problem to be graded. Consider the Markov chain whose transition matrix is

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 1/3 & 2/3 \end{bmatrix}.$$

- (a) What is the canonical form of the matrix P where Q is transient \rightarrow transient and R is transient \rightarrow absorbing states. (3 points)

- (b) What is the expected number of times the system will visit state 4 given it starts from state 2? Write out the N matrix. (3 points)

(c) If the system starts in state 3, what is the expected number of times in states 2, 3, and 4 before absorption? (2 points)

(d) Write the B matrix, the probability of absorption, where $B = NR$. Explain the meaning of the B matrix. (2 points)