

Math 21b Final Exam—Thursday, May 15, 2003—Solutions

1. (20 points) True or False. No justification is necessary, simply circle **T** or **F** for each statement.

T **F** (a) If W is a subspace of \mathbb{R}^n and \mathbf{x} is not in W , then $\mathbf{x} - \text{proj}_W \mathbf{x}$ is not zero.

Solution. True, since \mathbf{x} cannot be written as a linear combination of vectors in W .

T **F** (b) A linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is completely determined by what T does to each column of the $n \times n$ identity matrix.

Solution. True. , since T is determined by what it does to a basis and the columns of I_n form a basis for \mathbb{R}^n .

T **F** (c) If the matrix A^2 is diagonalizable, then A is diagonalizable.

Solution. False. If

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

then A^2 is the zero matrix which is diagonalizable.

T **F** (d) Similar matrices have the same eigenvectors.

Solution. False. If

$$A = \begin{bmatrix} 2 & -3 \\ 2 & -5 \end{bmatrix}$$

then

$$D = \begin{bmatrix} 1 & 0 \\ 0 & -4 \end{bmatrix} = \begin{bmatrix} 2/5 & -1/5 \\ -1/5 & 3/5 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 2 & -5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

is similar to A . However, $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector for A since $A\mathbf{v} = -4\mathbf{v}$, but

$$D\mathbf{v} = \begin{bmatrix} 1 \\ -8 \end{bmatrix}$$

is not a multiple of \mathbf{v} .

T F (e) If A is similar to B , then A^2 is similar to B^2 .

Solution. True. If $A = S^{-1}BS$, then

$$A^2 = (S^{-1}BS)^2 = S^{-1}B^2S.$$

T F (f) Let A be a 4×4 matrix with three distinct eigenvalues. If one of the eigenspaces is two dimensional, then A is diagonalizable.

Solution. True. Let $\lambda_1, \lambda_2, \lambda_3$ be three distinct eigenvalues of A with eigenspaces E_1, E_2, E_3 , respectively. The minimum dimension of each eigenspace is one, and one eigenspace, say E_1 , has dimension two. Therefore, A has an eigenbasis and is diagonalizable.

T F (g) There exists a 5×5 matrix with no real eigenvalues.

Solution. False. If A is characteristic polynomial, its characteristic polynomial must have degree five. Any polynomial of odd degree has at least one real root. Therefore, A must have a real eigenvalue.

T F (h) An orthogonal matrix is orthogonally diagonalizable.

Solution. False. The matrix

$$\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

is orthogonal, but not symmetric. Therefore, it is not orthogonally diagonalizable.

T F (i) If W_1 and W_2 are subspaces of a linear space V , then $W_1 \cup W_2$ is always a subspace of V . (The set $W_1 \cup W_2$ is the set of all elements in V that are in either W_1 or W_2 or in both W_1 and W_2 .)

Solution. False. If W_1 and W_2 are two distinct one-dimensional subspaces of \mathbb{R}^2 , then their union is a pair of distinct lines through the origin, which is not a subspace of \mathbb{R}^2 .

T F (j) If a real matrix A has only the eigenvalues 1 and -1 , then A is an orthogonal matrix.

Solution. False. The matrix

$$\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$$

is not orthogonal.

2. Let $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ be an orthonormal basis for \mathbb{R}^n , and let $\lambda_1, \dots, \lambda_n$ be any real scalars. Define A to be the $n \times n$ matrix

$$A = \lambda_1 \mathbf{u}_1 \mathbf{u}_1^T + \dots + \lambda_n \mathbf{u}_n \mathbf{u}_n^T.$$

- (a) Show that A is symmetric. (2 points)

Solution.

$$\begin{aligned} A^T &= (\lambda_1 \mathbf{u}_1 \mathbf{u}_1^T + \dots + \lambda_n \mathbf{u}_n \mathbf{u}_n^T)^T \\ &= \lambda_1 (\mathbf{u}_1 \mathbf{u}_1^T)^T + \dots + \lambda_n (\mathbf{u}_n \mathbf{u}_n^T)^T \\ &= \lambda_1 (\mathbf{u}_1^T)^T \mathbf{u}_1^T + \dots + \lambda_n (\mathbf{u}_n^T)^T \mathbf{u}_n^T \\ &= \lambda_1 \mathbf{u}_1 \mathbf{u}_1^T + \dots + \lambda_n \mathbf{u}_n \mathbf{u}_n^T \\ &= A. \end{aligned}$$

- (b) Show that $\lambda_1, \dots, \lambda_n$ are eigenvalues of A . (5 points)

Solution.

$$\begin{aligned} A\mathbf{u}_i &= (\lambda_1 \mathbf{u}_1 \mathbf{u}_1^T + \dots + \lambda_n \mathbf{u}_n \mathbf{u}_n^T)\mathbf{u}_i \\ &= \lambda_1 \mathbf{u}_1 \mathbf{u}_1^T \mathbf{u}_i + \dots + \lambda_i \mathbf{u}_i \mathbf{u}_i^T \mathbf{u}_i + \dots + \lambda_n \mathbf{u}_n \mathbf{u}_n^T \mathbf{u}_i \\ &= \lambda_i \mathbf{u}_i. \end{aligned}$$

3. Let

$$A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}.$$

- (a) Find an eigenbasis for A . [*Hint*: One of the eigenvalues of A is 7.] (4 points)

Solution. Since the characteristic polynomial of A is

$$\det(\lambda I - A) = \lambda^3 - 12\lambda^2 + 21\lambda + 98 = (\lambda - 7)^2(\lambda + 2),$$

the eigenvalues of A are $\lambda_1 = 7$ and $\lambda_2 = -2$. The kernel of the matrix $\lambda_1 I - A$ has a basis

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix}.$$

The kernel of the matrix $\lambda_2 I - A$ has a basis

$$\mathbf{v}_3 = \begin{bmatrix} -1 \\ -1/2 \\ 1 \end{bmatrix}.$$

Thus,

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -1 \\ -1/2 \\ 1 \end{bmatrix}$$

is an eigenbasis for A .

- (b) Orthogonally diagonalize A (5 points)

Solution. If we apply the Gram-Schmidt orthogonalization process to $\mathbf{v}_1, \mathbf{v}_2$, we find that

$$\mathbf{u}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} -1/\sqrt{18} \\ 4/\sqrt{18} \\ 1/\sqrt{18} \end{bmatrix}$$

is an orthonormal basis for E_7 . Similarly, E_{-2} has orthonormal basis

$$\mathbf{u}_3 = \begin{bmatrix} -2/3 \\ -1/3 \\ 2/3 \end{bmatrix}.$$

Thus, $A = PDP^T$, where

$$D = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -2 \end{bmatrix} \text{ and } P = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{18} & -2/3 \\ 0 & 4/\sqrt{18} & -1/3 \\ 1/\sqrt{2} & 1/\sqrt{18} & 2/3 \end{bmatrix}.$$

4. (a) Let A be an $m \times n$ matrix such that $A^T A$ is invertible. Show that the columns of A are linearly independent. Be careful, A may not be a square matrix. (5 points)

Solution. If

$$c_1 \mathbf{a}_1 + \cdots + c_n \mathbf{a}_n = \mathbf{0},$$

we must show $c_1 = \cdots = c_n = 0$. However,

$$\mathbf{0} = A^T \mathbf{0} = A^T (c_1 \mathbf{a}_1 + \cdots + c_n \mathbf{a}_n) = c_1 A^T \mathbf{a}_1 + \cdots + c_n A^T \mathbf{a}_n$$

and $A^T \mathbf{a}_1, \dots, A^T \mathbf{a}_n$ are exactly the columns of $A^T A$. Since $A^T A$ is invertible, these vectors are linearly independent. Therefore, $c_1 = \cdots = c_n = 0$.

- (b) Show that the matrix

$$B = \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix}$$

is invertible if and only if a , b , and c are distinct real numbers. *You do not need to use part (a) to solve this problem.* (4 points)

Solution.

$$\begin{aligned} \det(B) &= \det \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix} \\ &= \det \begin{bmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{bmatrix} \\ &= (b-a)(c^2-a^2) - (c-a)(b^2-a^2) \\ &= (b-a)(c-a)(c-b). \end{aligned}$$

This determinant is nonzero if and only if a , b , and c are distinct real numbers.

5. We wish to find the function

$$y = c_1x + c_2 \cos\left(\frac{\pi x}{2}\right)$$

that best fits the following data using least-squares.

$$\begin{array}{c|cccc} x & 0 & 1 & 2 & 3 \\ \hline y & 0 & 2 & 2 & 4 \end{array}$$

(a) Write the matrix equation that must be solved. (4 points)

Solution. $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 2 & -1 \\ 3 & 0 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 0 \\ 2 \\ 2 \\ 4 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

(b) Find the least-squares solution and write the corresponding function $f(x)$. (4 points)

Solution. We must solve the normal equations $A^T A\mathbf{x} = A^T \mathbf{b}$. However,

$$\mathbf{x} = (A^T A)^{-1} A^T \mathbf{b} = \begin{bmatrix} 4/3 \\ 1/3 \end{bmatrix}.$$

Thus,

$$f(x) = \frac{4}{3}x + \frac{1}{3} \cos\left(\frac{\pi x}{2}\right).$$

6. Consider the dynamical system

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x},$$

where

$$A = \begin{bmatrix} -3 & 2 \\ -1 & -1 \end{bmatrix}$$

- (a) Find the solution of $d\mathbf{x}/dt = A\mathbf{x}$ with the initial condition

$$\mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Be sure to state your solution in terms of real numbers and real functions and not the complex exponential function. (5 points)

Solution. The characteristic polynomial of A is

$$\det(\lambda I - A) = \lambda^2 + 4\lambda + 5$$

and has roots $\lambda = -2 \pm i$. For the eigenvalue $\lambda = -2 + i$, we can find an eigenvector

$$\mathbf{v} + i\mathbf{w} = \begin{bmatrix} 1 - i \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + i \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

Therefore, the solution to our dynamical system is

$$\mathbf{x}(t) = e^{-2t} S \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} S^{-1} \mathbf{x}_0 = e^{-2t} \begin{bmatrix} \cos t + \sin t \\ \cos t \end{bmatrix},$$

where

$$S = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}.$$

- (b) Sketch the graph of the trajectory for a solution of $d\mathbf{x}/dt = A\mathbf{x}$. (2 points)

Solution. The trajectory spirals in beginning at the point $(1, 1)$.

- (c) In the redwood forests of northern California dusky-footed wood rats provide up to 80% of the diet of the spotted owl, the main predator of the wood rat. Denote the owl and wood rat populations at time k by $\mathbf{x}_k = [O_k, R_k]^T$, where O_k is the number of owls in the region and R_k is the number of rats (measured in thousands). Suppose that the owl and rat population evolve according to the discrete dynamical system

$$\begin{aligned} O_{k+1} &= 0.5O_k + 0.4R_k \\ R_{k+1} &= -pO_k + 1.1R_k, \end{aligned}$$

where $p > 0$ is a parameter to be specified. If $p = 0.104$, the eigenvalues of the matrix for the matrix of the dynamical system are $\lambda_1 = 1.02$ and $\lambda_2 = 0.58$. The eigenvectors for λ_1 and λ_2 are

$$\mathbf{v}_1 = \begin{bmatrix} 10 \\ 13 \end{bmatrix} \text{ and } \mathbf{v}_2 = \begin{bmatrix} 5 \\ 1 \end{bmatrix},$$

respectively. What is the long-term monthly growth (or decay) rate of the owl and wood rat populations. Be specific and carefully justify your conclusion. (4 points)

Solution. Any initial population vector \mathbf{x}_0 can be written as $\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$. For $k \geq 0$,

$$\mathbf{x}_k = A^k \mathbf{x}_0 = c_1(1.02)^k \mathbf{v}_1 + c_2(0.58)^k \mathbf{v}_2,$$

and $(0.58)^k \rightarrow 0$ very quickly as k gets large. Thus,

$$\mathbf{x}_k \approx c_1(1.02)^k \begin{bmatrix} 10 \\ 13 \end{bmatrix}$$

for sufficiently large k , and the owl and wood rat populations grow at a rate of approximately 2% per month.

7. (a) Let $\mathbf{u} = \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix}$ be a unit vector in \mathbb{R}^2 . Give a 2×2 matrix R_θ which represents reflection along the line spanned by \mathbf{u} . (4 points)

Solution. The reflection of any vector \mathbf{v} along the line spanned by \mathbf{u} is given by

$$R_\theta(\mathbf{v}) = 2 \text{proj}_{\mathbf{u}} \mathbf{v} - \mathbf{v} = 2(\mathbf{u} \cdot \mathbf{v})\mathbf{u} - \mathbf{v}.$$

To find the matrix of this transformation, we compute

$$R_\theta(\mathbf{e}_1) = 2(\mathbf{u} \cdot \mathbf{e}_1)\mathbf{u} - \mathbf{e}_1 = \begin{bmatrix} 2 \cos^2 \theta - 1 \\ 2 \sin \theta \cos \theta \end{bmatrix} = \begin{bmatrix} \cos 2\theta \\ \sin 2\theta \end{bmatrix}$$

and

$$R_\theta(\mathbf{e}_2) = 2(\mathbf{u} \cdot \mathbf{e}_2)\mathbf{u} - \mathbf{e}_2 = \begin{bmatrix} 2 \sin \theta \cos \theta \\ 2 \sin^2 \theta - 1 \end{bmatrix} = \begin{bmatrix} \sin 2\theta \\ -\cos 2\theta \end{bmatrix}.$$

Therefore, the matrix of our transformation is

$$\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}.$$

(b) Let

$$\mathbf{u}_1 = \begin{bmatrix} \cos \theta_1 \\ \sin \theta_1 \end{bmatrix} \text{ and } \mathbf{u}_2 = \begin{bmatrix} \cos \theta_2 \\ \sin \theta_2 \end{bmatrix}$$

be two unit vectors in \mathbb{R}^2 . Show that the composition matrix $R_{\theta_2}R_{\theta_1}$ is a counterclockwise rotation by some angle. What is this angle in terms of θ_1 and θ_2 ? (4 points)

Solution. Since

$$\begin{aligned} R_{\theta_2}R_{\theta_1} &= \begin{bmatrix} \cos 2\theta_2 & \sin 2\theta_2 \\ \sin 2\theta_2 & -\cos 2\theta_2 \end{bmatrix} \begin{bmatrix} \cos 2\theta_1 & \sin 2\theta_1 \\ \sin 2\theta_1 & -\cos 2\theta_1 \end{bmatrix} \\ &= \begin{bmatrix} \cos 2\theta_1 \cos 2\theta_2 + \sin 2\theta_1 \sin 2\theta_2 & \cos 2\theta_1 \sin 2\theta_2 - \sin 2\theta_1 \cos 2\theta_2 \\ -(\cos 2\theta_2 \sin 2\theta_1 - \sin 2\theta_2 \cos 2\theta_1) & \cos 2\theta_1 \cos 2\theta_2 + \sin 2\theta_1 \sin 2\theta_2 \end{bmatrix} \\ &= \begin{bmatrix} \cos 2(\theta_2 - \theta_1) & -\sin 2(\theta_2 - \theta_1) \\ \sin 2(\theta_2 - \theta_1) & \cos 2(\theta_2 - \theta_1) \end{bmatrix}, \end{aligned}$$

we have a counterclockwise rotation by $2(\theta_2 - \theta_1)$.

(c) Now consider an arbitrary orthogonal $n \times n$ matrix A . Show that $\det(A) = \pm 1$. If $\det(A) = +1$, we call A a rotation. If $\det(A) = -1$, we call A a reflection. Show that the product of any two reflections is a rotation. *This part does not require any complicated calculations.* (2 points)

Solution. Since A is orthogonal, $A^T A = I$. Thus,

$$1 = \det(A^T A) = \det(A^T) \det(A) = (\det(A))^2,$$

and $\det(A) = \pm 1$. If A is a reflection, then $\det(A) = -1$ and $\det(A^2) = (\det(A))^2 = 1$. Hence, A^2 is a rotation.

8. The *trace* of a $n \times n$ matrix is the sum of its diagonal elements.

- (a) Show that the trace of FG is equal to the trace of GF for two $n \times n$ matrices F and G . (4 points)

Solution. If $C = FG$, then the diagonal elements of C are

$$c_{ii} = \sum_{k=1}^n f_{ik}g_{ki}$$

and the trace of FG is

$$\sum_{i=1}^n c_{ii} = \sum_{i=1}^n \sum_{k=1}^n f_{ik}g_{ki}$$

Similarly, if $D = GF$, the trace of D is

$$\sum_{k=1}^n d_{kk} = \sum_{k=1}^n \sum_{i=1}^n g_{ki}f_{ik}.$$

Since

$$\sum_{i=1}^n c_{ii} = \sum_{i=1}^n \sum_{k=1}^n f_{ik}g_{ki} = \sum_{k=1}^n \sum_{i=1}^n g_{ki}f_{ik} = \sum_{k=1}^n d_{kk},$$

the trace of FG is equal to the trace of GF .

- (b) If A and B are similar matrices, show that the trace of A is equal to the trace of B . (4 points)

Solution. If $B = S^{-1}AS$, then

$$\text{Tr}(B) = \text{Tr}(S^{-1}AS) = \text{Tr}(S^{-1}SA) = \text{Tr}(A).$$

9. Let U be the subspace of \mathbf{R}^4 defined by the equations

$$\begin{aligned} 2x_1 + x_2 + x_3 + x_4 &= 0 \\ x_1 - x_2 + x_3 - x_4 &= 0, \end{aligned}$$

and let V be the subspace of \mathbf{R}^4 defined by the equations

$$\begin{aligned} -x_1 + x_2 + 2x_3 - 2x_4 &= 0 \\ x_1 + 2x_2 + 2x_4 &= 0. \end{aligned}$$

- (a) Find a basis for $U \cap V$, where $U \cap V$ is the set of vectors in \mathbb{R}^4 that are in both U and V . (4 points)

Solution. The subspace $U \cap V$ is the set of all solutions to the system

$$\begin{array}{ccccrc} 2x_1 & +x_2 & +x_3 & +x_4 & = & 0 \\ x_1 & -x_2 & +x_3 & -x_4 & = & 0 \\ -x_1 & +x_2 & +2x_3 & -2x_4 & = & 0 \\ x_1 & +2x_2 & & +2x_4 & = & 0. \end{array}$$

The reduced row echelon form of the matrix of this system is

$$\begin{bmatrix} 1 & 0 & 0 & 2/3 \\ 0 & 1 & 0 & 2/3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and a basis for the kernel of this matrix is given by

$$\begin{bmatrix} -2/3 \\ -2/3 \\ 1 \\ 1 \end{bmatrix} \text{ or } \begin{bmatrix} -2 \\ -2 \\ 3 \\ 3 \end{bmatrix}$$

- (b) Find a basis for $(U \cap V)^\perp$. (4 points)

Solution. A basis for $(U \cap V)^\perp$ can be found by computing

$$\ker(A^T) = \text{image}(A)^\perp,$$

where

$$A = \begin{bmatrix} -2 \\ -2 \\ 3 \\ 3 \end{bmatrix}.$$

The reduced row echelon form of A^T is

$$[1 \quad 1 \quad -3/2 \quad -3/2].$$

Therefore,

$$\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3/2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3/2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

is a basis for $(U \cap V)^\perp$.

10. **You only need to do Problem 10 or Problem 11. Credit will be given for only one of these two problems. Circle Problem 10 on the first page if you wish this problem to be graded.**

- (a) Solve the equation $x'' - 3x' + 2x = 0$, if $x(0) = 1$ and $x'(0) = -1$. (4 points)

Solution. The characteristic equation of $x'' - 3x' + 2x = 0$ is

$$r^2 - 3r + 2 = (r - 2)(r - 1) = 0.$$

Thus, the general solution is

$$x(t) = c_1 e^{2t} + c_2 e^t.$$

Since

$$x'(t) = 2c_1 e^{2t} + c_2 e^t$$

we can apply the initial conditions $x(0) = 1$ and $x'(0) = -1$ to obtain the system

$$\begin{aligned} c_1 + c_2 &= 1 \\ 2c_1 + c_2 &= -1. \end{aligned}$$

Solving this system, we find that $c_1 = -2$ and $c_2 = 3$; therefore, the solution that we are seeking is

$$x(t) = -2e^{2t} + 3e^t.$$

- (b) Consider the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}.$$

Show that

$$u(x, t) = b_n e^{-n^2 t} \sin nx$$

is a solution satisfying the boundary conditions $u(0, t) = u(\pi, t) = 0$ for $n = 1, 2, 3, \dots$, where b_n is an arbitrary constant. (3 points)

Solution. The function $u(x, t) = b_n e^{-n^2 t} \sin nx$ satisfies the boundary conditions, since

$$\begin{aligned} u(0, t) &= b_n e^{-n^2 t} \sin 0 = 0 \\ u(\pi, t) &= b_n e^{-n^2 t} \sin n\pi = 0. \end{aligned}$$

To show that u also satisfies

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},$$

we only need to observe that

$$\begin{aligned} \frac{\partial u}{\partial t} &= -n^2 b_n e^{-n^2 t} \sin nx \\ \frac{\partial^2 u}{\partial x^2} &= -n^2 b_n e^{-n^2 t} \sin nx. \end{aligned}$$

(c) Using the initial condition,

$$u(x, 0) = f(x) = 80 \sin^3 x = 60 \sin x - 20 \sin 3x.$$

solve the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

with boundary conditions $u(0, t) = u(\pi, t) = 0$. (3 points)

Solution. By the principle of superposition,

$$u(x, t) = \sum_{n=1}^{\infty} b_n e^{-n^2 t} \sin nx$$

is a solution to

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

with boundary conditions $u(0, t) = u(\pi, t) = 0$. Applying the initial condition $u(x, 0) = 60 \sin x - 20 \sin 3x$, we know that

$$u(x, 0) = \sum_{n=1}^{\infty} b_n \sin nx = 60 \sin x - 20 \sin 3x.$$

Thus, $b_1 = 60$ and $b_3 = -20$ with all other constants vanishing. Therefore, our solution is

$$u(x, t) = 60e^{-t} \sin x - 20e^{-9t} \sin 3x.$$

11. **You only need to do Problem 10 or Problem 11. Credit will be given for only one of these two problems. Circle Problem 11 on the first page if you wish this problem to be graded.** Consider the Markov chain whose transition matrix is

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 1/3 & 2/3 \end{bmatrix}.$$

- (a) What is the canonical form of the matrix P where Q is transient \rightarrow transient and R is transient \rightarrow absorbing states. (3 points)

Solution.

$$\begin{bmatrix} 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/3 & 2/3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

- (b) What is the expected number of times the system will visit state 4 given it starts from state 2? Write out the N matrix. (3 points)

Solution.

$$I - Q = \begin{bmatrix} 1 & -1/2 & 0 \\ -1/2 & 1 & -1/2 \\ 0 & -1/3 & 1/3 \end{bmatrix}$$

and

$$N = (I - Q)^{-1} \begin{bmatrix} 2 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 4 & 9 \end{bmatrix}.$$

- (c) If the system starts in state 3, what is the expected number of times in states 2, 3, and 4 before absorption? (2 points)

Solution. If the system starts in state 3, then the expected number of times it visits state 1, 2, 3 before absorption is 2, 4, 9.

- (d) Write the B matrix, the probability of absorption, where $B = NR$. Explain the meaning of the B matrix. (2 points)

Solution.

$$B = \begin{bmatrix} 2 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 4 & 9 \end{bmatrix} \begin{bmatrix} 1/2 \\ 0 \\ 0 \end{bmatrix}$$

The probability of absorption is one starting from any state.