MATH 21b FALL 2001

EXAMINATION 1

Tuesday, 23 October, 2001.

Name: ____

Teaching Fellow (PLEASE CIRCLE)

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Instructions:

1. Do not open this test until told to do so.
2. Please do not detach any pages from this exam.
3. You may use your calculator and one (1) page of notes not exceeding 8.5 by 11 inches in size.
4. You may use the backs of test sheets for scratch paper, or to continue your working on problems. If you write on the backs of the test sheets, **please label your working very clearly**.
5. SHOW ALL YOUR WORK.
6. Exam proctors are not permitted to answer questions regarding the content of the test.
7. Many of the questions have precisely worded instructions. **Be sure to read all instructions carefully,** and do all that is asked.
8. Please try your best - try to relax and show us what you can do!

Problem	Total	Score
1	19	
2	20	
3	10	
4	16	
5	13	
6	12	
7	10	
Total	100	

1. (19 points total)

Let $T: \mathbf{R}^3 \to \mathbf{R}^4$ be the linear transformation defined by the matrix A given below.

$$A = \begin{bmatrix} 1 & -2 & 5 \\ 2 & 5 & -8 \\ -1 & -4 & 7 \\ 3 & 1 & 1 \end{bmatrix}$$

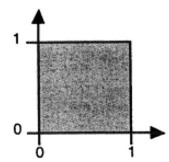
(a) **(10 points)** Find a basis for the *kernel* of T. Be careful to explain your reasoning and show your work.

(b) **(5 points)** Find a basis for the *image* of T. Be careful to explain your reasoning and show your work.

(c) **(4 points)** What is the rank of the matrix A ?

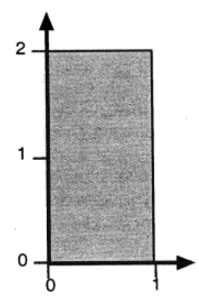
2. (20 points total)

Each of the diagrams shown below is obtained from the unit square:

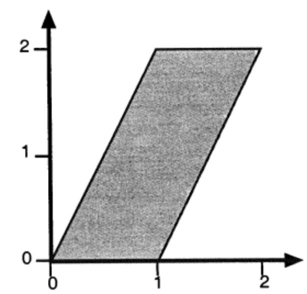


by some kind of transformation T: $\mathbf{R}^2 \rightarrow \mathbf{R}^2$. For each of the diagrams shown below, decide whether or not the transformation was a linear transformation or not. If you believe that the transformation was a linear transformation, find a matrix that could represent that transformation. If you believe that the transformation was not a linear transformation, explain why not.

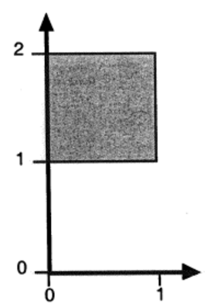
(a) (5 points)



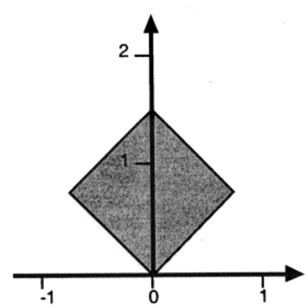
(b) (5 points)



(c) (5 points)



(d) (5 points)



3. (10 points total)

In this problem, A and B are 2 by 2 matrices. The **inverses** of A and B are given below.

$$A^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad B^{-1} = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

(a) (5 points) Solve the system of linear equations:

$$B\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

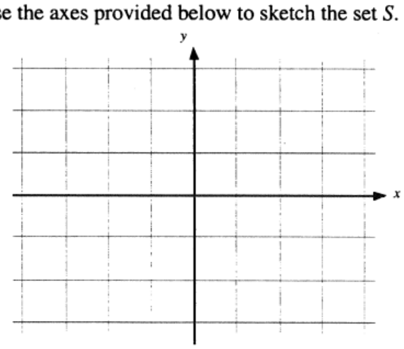
(b) (5 points) Find the 2 by 2 matrix: $(BA)^{-1}$.

4. (16 points total)

Let S be the set of all vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ whose components satisfy *either*:

$$2x - y = 0 \qquad or \qquad 3x + 5y = 0.$$

(a) **(4 points)** Use the axes provided below to sketch the set S.



(b) **(6 points)** Is S a subspace of the vector space $\mathbf{R}^2$? Be careful to note what properties a subset must have in order to be a subspace, and use these properties to justify your answer.

(c) **(6 points)** Does S have any subsets that are subspaces of $\mathbf{R}^2$? If not, explain why not. If so, give an example of such a subset.

5. (13 points total)

The symbols: P_2 denote the vector space of polynomials that have degree at most equal to 2. In this problem, $T:\mathbf{R}^2 \to P_2$ will always denote a linear transformation with the following properties:

$$T\left(\begin{bmatrix} -1 \\ 4 \end{bmatrix}\right) = t^2 - 3 \quad \text{and} \quad T\left(\begin{bmatrix} -2 \\ 9 \end{bmatrix}\right) = t + 1.$$

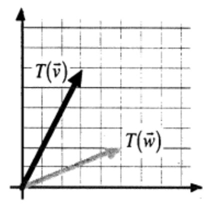
(a) **(5 points)** Calculate: $T\left(\begin{bmatrix} 7 \\ -2 \end{bmatrix}\right)$.

(b) **(5 points)** Find a vector $\begin{bmatrix} a \\ b \end{bmatrix}$ so that $T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = 3t^2 + 4t - 5$.

(c) **(3 points)** Find a quadratic polynomial that is ***not*** in the image of T.

6. (12 points total)

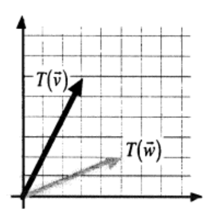
Let T: $\mathbf{R}^2 \rightarrow \mathbf{R}^2$ be a linear transformation. The diagram given below shows the result of applying this linear transformations to the vectors: $\vec{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$



(a) **(4 points)** Find a basis for the *image* of the linear transformation T.

(b) **(4 points)** Express $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$ as a linear combination of the basis vectors that you found in Part (a).

(c) **(4 points)** Using the axes provided, sketch the vector: $T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$.



7. (10 points total)

How many linear transformations T: $\mathbf{R}^2 \to \mathbf{R}^2$ have the property of being their own inverse? Whatever your final answer is, make sure that you justify it.

Hint: One way (but not the only way) to approach this problem is to determine how many linear transformations T: $\mathbf{R}^2 \to \mathbf{R}^2$ satisfy the equation:

$$T(T(\vec{x})) = \vec{x}$$

for every single vector $\vec{x}$ in $\mathbf{R}^2$.