Math 21b Midterm 1 Solutions - Fall 2001

1. (a) Simple calculation shows that

$$rref(A) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus for $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ in the kernel of T, we know $x_1 = -x_3$ and $x_2 = 2x_3$. Thus the kernel of T is spanned by $\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$, and in fact this is a basis for the kernel since this spanning set is clearly linearly independent.

(b) The leading 1's in $rref(A)$ were in the first and second columns, so the first and second columns of A span the image of T. These are clearly linearly independent (since one is not a multiple of the other), so they form a basis for the image of T:

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 5 \\ -4 \\ 1 \end{bmatrix} \right\}.$$

(c) $rank(A) = \dim(im(A)) = 2.$

2. (a) Linear: $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$.

(b) Linear: $\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$.

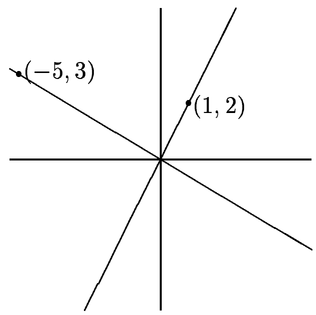
(c) Not linear because $T(0) = 1$ (not 0).

(d) Linear: $\begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$.

3. (a) $B \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Longrightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = B^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 11 \\ 15 \end{bmatrix}.$

(b) $(BA)^{-1} = A^{-1}B^{-1} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}.$

4. (a) The set S is sketched on the axes below:



(b) S is not a subspace of the vector space $\mathbb{R}^2$ because it is not closed under addition. For example, $\begin{bmatrix}1\\2\end{bmatrix}+\begin{bmatrix}5\\-3\end{bmatrix}=\begin{bmatrix}6\\-1\end{bmatrix}$, and $\begin{bmatrix}6\\-1\end{bmatrix}$ does not satisfy either of the given equations, but $\begin{bmatrix}1\\2\end{bmatrix}$ satisfies the first one and $\begin{bmatrix}5\\-3\end{bmatrix}$ satisfies the second one.

(c) S has three subsets that are subspaces of $\mathbb{R}^2$. They are

i. The vectors satisfying $2x - y = 0$.

ii. The vectors satisfying $3x + 5y = 0$.

iii. The origin.

5. (a) We want to find a and b such that $a\begin{bmatrix}-1\\4\end{bmatrix}+b\begin{bmatrix}-2\\9\end{bmatrix}=\begin{bmatrix}7\\-2\end{bmatrix}$. Solving the system yields $a = -59$ and $b = 26$. Thus $T\left(\begin{bmatrix}7\\-2\end{bmatrix}\right) = -59(t^2-3)+26(t+1) = -59t^2+26t+203$.

(b) $3t^2+4t-5 = 3(t^2-3)+4(t+1)$. Thus

$$\begin{bmatrix}a\\b\end{bmatrix}=3\begin{bmatrix}-1\\4\end{bmatrix}+4\begin{bmatrix}-2\\9\end{bmatrix}=\begin{bmatrix}-11\\48\end{bmatrix}.$$

(c) The quadratic t^2 is not in the image of T because there is no linear combination of t^2-3 and $t+1$ that equals t^2.

6. (a) The diagram given shows that $T(\vec{v})$ and $T(\vec{w})$ are linearly independent. And since $\dim(im(T)) \leq 2$, the vectors $T(\vec{v})$ and $T(\vec{w})$ must form a basis for the image of T.

(b) Since T is linear, $T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = T(\vec{w}) - T(\vec{v})$.

(c) Again since T is linear, $T\left(\begin{bmatrix}1\\2\end{bmatrix}\right) = T(\vec{v}) + T(\vec{w})$. This can be drawn on the axis provided by adding the two vectors to get a vector whose tip is at approximately $(8, 8)$ if we consider the grid marks to be the units.

7. The linear transformation T is from $\mathbb{R}^2$ to $\mathbb{R}^2$, so the matrix A, representing T, is a 2 x 2 matrix. Recall that $\det(A) = (\det(A^{-1}))^{-1}$. Since $A = A^{-1}$, this means $\det(A) = \pm 1$. If $\det(A) = 1$, then

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = A^{-1}.$$

Thus we get that $a = d$, $b = c = 0$, and $ad = 1$. Therefore the only possibilities for A are $\pm I_2$. However, if $\det(A) = -1$, then A must have the form:

$$A = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$$

where $-a^2 - bc = -1$. But notice that this equation has infinitely many solutions, since we can rewrite it as:

$$-bc = a^2 - 1 = (a-1)(a+1),$$

so any set of values $(a, b, c) = (a, 1-a, 1+a)$ will be valid. Thus any matrix of the form

$$A = \begin{bmatrix} a & 1-a \\ 1+a & -a \end{bmatrix}$$

is its own inverse. Thus there are infinitely many such matrices, and so infinitely many self-inverting linear transformations.

(Note: this does not characterize all matrices whose square is the identity, but it sufficiently shows that there are infinitely many.)