Last Name: SOLUTIONS

First Name: ____________

# Mathematics 21b

First Midterm Examination
October 25, 1999

**Your Section (circle one):**

| John Boller | Noam Elkies |
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| MWF 10 | MWF 11 |

| Question | Points | Score |
|---|---|---|
| 1 | 10 | |
| 2 | 10 | |
| 3 | 10 | |
| 4 | 10 | |
| Total | 40 | |

**No calculators are allowed.**

1. Consider the system of linear equations

$$\begin{cases} 3x_3 - 2x_2 = -3, \\ x_3 + 2x_4 + 4x_1 = 1, \\ 2x_1 + x_2 - x_3 + x_4 = 2. \end{cases}$$

(a) (2 pts.) Write the coefficient matrix and augmented matrix for this system.

$$A = \begin{bmatrix} 0 & -2 & 3 & 0 \\ 4 & 0 & 1 & 2 \\ 2 & 1 & -1 & 1 \end{bmatrix} \qquad [A \,\vdots\, \vec{b}] = \left[\begin{array}{cccc:c} 0 & -2 & 3 & 0 & -3 \\ 4 & 0 & 1 & 2 & 1 \\ 2 & 1 & -1 & 1 & 2 \end{array}\right]$$

(b) (5 pts.) Calculate the row-reduced echelon form of the augmented matrix.

$$\left[\begin{array}{cccc:c} 0 & -2 & 3 & 0 & -3 \\ 4 & 0 & 1 & 2 & 1 \\ 2 & 1 & -1 & 1 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{cccc:c} 1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 1 \\ 4 & 0 & 1 & 2 & 1 \\ 0 & -2 & 3 & 0 & -3 \end{array}\right] \longrightarrow \left[\begin{array}{cccc:c} 1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 1 \\ 0 & -2 & 3 & 0 & -3 \\ 0 & -2 & 3 & 0 & -3 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cccc:c} 1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 1 \\ 0 & 1 & -\frac{3}{2} & 0 & \frac{3}{2} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cccc:c} 1 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 1 & -\frac{3}{2} & 0 & \frac{3}{2} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(c) (3 pts.) Find the general solution of the linear system. Verify that your answer does in fact satisfy all the equations.

Let $x_4 = t$

Let $x_3 = s$

Then $x_2 = \frac{3}{2} + \frac{3}{2}s$

$x_1 = \frac{1}{4} - \frac{1}{4}s - \frac{1}{2}t$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} - \frac{1}{4}s - \frac{1}{2}t \\ \frac{3}{2} + \frac{3}{2}s \\ s \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ \frac{3}{2} \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -\frac{1}{4} \\ \frac{3}{2} \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -\frac{1}{2} \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Eq. 1: $3(s) - 2\left(\frac{3}{2} + \frac{3}{2}s\right) = 3s - 3 - 3s = -3$

Eq. 2: $s + 2t + 4\left(\frac{1}{4} - \frac{1}{4}s - \frac{1}{2}t\right) = s + 2t + 1 - s - 2t = 1$

Eq. 3: $2\left(\frac{1}{4} - \frac{1}{4}s - \frac{1}{2}t\right) + \left(\frac{3}{2} + \frac{3}{2}s\right) - s + t$

$= \frac{1}{2} - \frac{1}{2}s - t + \frac{3}{2} + \frac{3}{2}s - s + t$

$= 2$

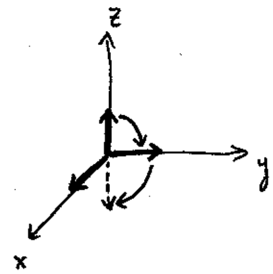
2. Let $A$ and $B$ be the matrices

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}.$$

(a) (4 pts.) Describe $A, B$ geometrically as linear transformations.

$A\vec{e}_1 = \vec{e}_1$
$A\vec{e}_2 = 0$
$A\vec{e}_3 = \vec{e}_3$

$B\vec{e}_1 = \vec{e}_1$
$B\vec{e}_2 = -\vec{e}_3$
$B\vec{e}_3 = \vec{e}_2$




$A$ is projection onto the $xz$-plane

$B$ is rotation by $\frac{\pi}{2}$ clockwise in the $yz$-plane.

(b) (3 pts.) What are the ranks of $A$ and $B$? Is either $A$ or $B$ invertible? Justify your answers.

$\text{rank}(A) = 2$ since $\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ and hence $\dim(\ker(A)) = 1$ and $A$ is not invertible.

$\text{rank}(B) = 3$ since $\text{rref}(B) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and hence $\dim(\ker(B)) = 0$ and $B$ is invertible.

(The inverse of a rotation by $\theta$ is a rotation by $-\theta$.)

(c) (3 pts.) Do $A$ and $B$ commute? Interpret your result geometrically.

$$AB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \qquad BA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad AB \neq BA$$

$(AB)\vec{e}_1 = \vec{e}_1$
$(AB)\vec{e}_2 = -\vec{e}_3$
$(AB)\vec{e}_3 = 0$

$(BA)\vec{e}_1 = \vec{e}_1$
$(BA)\vec{e}_2 = 0$
$(BA)\vec{e}_3 = \vec{e}_2$

Image $(AB) =$ $xz$-plane
Ker $(AB) =$ $z$-axis

Image $(BA) =$ $xy$-plane
Ker $(BA) =$ $y$-axis

3. (a) (1 pt. each) Define: If $T: \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation with matrix $A$, then

- kernel $\quad Ker(A) = \{ \vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{0} \}$

- image $\quad Im(A) = \{ y \in \mathbb{R}^m \mid A\vec{x} = \vec{y}$ for some $\vec{x} \in \mathbb{R}^n \}$

- rank $\quad rank(A) = \#$ leading 1's in rref(A)

- span $\quad$ The span of the set of vectors $\{\vec{v}_1, \dots, \vec{v}_n\}$ is the set of all possible linear combinations of those vectors $= \{ c_1\vec{v}_1 + \dots + c_n\vec{v}_n \mid c_1, \dots, c_n$ are scalars $\}$

- basis $\quad$ If $V$ is a subspace of $\mathbb{R}^n$ and $\vec{v}_1, \dots, \vec{v}_m \in V$, then $\{\vec{v}_1, \dots, \vec{v}_m\}$ is a basis for $V$ if the $v_i$'s are linearly independent and the $v_i$'s span $V$.

(b) If $A$ is a $4 \times 5$ matrix of rank 4,

- (1 pt.) Is $A$ invertible? Why?

  No, only square matrices may be invertible.

- (1 pt.) What can you say about $\ker(A)$?

  $\dim(\ker(A)) = 1$.

- (1 pt.) What can you say about $\mathrm{im}(A)$?

  $\dim(\mathrm{im}(A)) = 4$

- (2 pts.) What can you say about a linear system whose coefficient matrix is $A$?

  The solution set will be a line.

4. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be orthogonal projection to the subspace of $\mathbb{R}^3$ with basis consisting of the single vector $\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$. [Note: this is not a unit vector!]

Let $\vec{w} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$.

Then for any $\vec{v} \in \mathbb{R}^3$, $\text{Proj}_{\vec{w}} \vec{v} = \dfrac{\vec{v} \cdot \vec{w}}{|\vec{w}|^2} \vec{w} = \dfrac{\vec{v} \cdot \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}}{5} \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$

(a) (4 pts.) Calculate $T\vec{e}_1$, $T\vec{e}_2$, $T\vec{e}_3$.

$$T(\vec{e}_1) = \frac{\vec{e}_1 \cdot \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}}{5} \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = \frac{2}{5}\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{4}{5} \\ -\frac{2}{5} \\ 0 \end{bmatrix}$$

$$T(\vec{e}_2) = \frac{\vec{e}_2 \cdot \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}}{5} \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = -\frac{1}{5}\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{2}{5} \\ \frac{1}{5} \\ 0 \end{bmatrix}$$

$$T(\vec{e}_3) = \frac{\vec{e}_3 \cdot \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}}{5} \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = 0\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(b) (2 pts.) Find the matrix for $T$.

$$A = \begin{bmatrix} \frac{4}{5} & -\frac{2}{5} & 0 \\ -\frac{2}{5} & \frac{1}{5} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(c) (4 pts.) What are the dimensions of $\ker(T)$ and $\text{im}(T)$? Find bases for these subspaces.

$$\text{Im}(T) = \text{span}\left( \begin{bmatrix} 4/5 \\ -2/5 \\ 0 \end{bmatrix}, \begin{bmatrix} -2/5 \\ 1/5 \\ 0 \end{bmatrix} \right)$$

$$\text{rref}(A) = \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

pivot

$$\text{Im}(T) = \text{span}\left( \begin{bmatrix} 4/5 \\ -2/5 \\ 0 \end{bmatrix} \right)$$

$\dim(\text{Im}(T)) = 1$

basis for $\text{Im}(T)$ is $\left\{ \begin{bmatrix} 4/5 \\ -2/5 \\ 0 \end{bmatrix} \right\}$.

$$\text{rref}(A) = \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 - \frac{1}{2}\cancel{x_2} = 0$

Let $x_3 = t$

Let $x_2 = s$

Then $x_1 = \frac{1}{2}s$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}s \\ s \\ t \end{bmatrix} = s\begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$\dim(\ker(T)) = 2$

basis for $\ker(T) = \left\{ \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

You must justify your answers to receive full credit.