Name:__________________________________

Math 21b Midterm I—Thursday, March 6, 2003

Please circle your section:

Thomas Judson Eduardo Saverin (CA) Jakob Topp (CA) MWF 10–11	Thomas Judson Rene Shen (CA) Albert Wang (CA) MWF 11–12	Kalle Karu Mark Bandstra (CA) MWF 12-1

Ken Chung Nathan Lange (CA) TuTh 10–11:30	Spiro Karigiannis Jeff Berton (CA) TuTh 11:30–1

Problem Number	Possible Points	Score
1	12	
2	10	
3	12	
4	12	
5	10	
6	11	
7	11	
8	11	
9	11	
Total	100	

Directions—Please Read Carefully! You have two hours to take this midterm. Pace yourself by keeping track of how many problems you have left to go and how much time remains. you do not have to answer the problems in any particular order, so move to another problem if you find you are stuck or spending too much time on a single problem. To receive full credit on a problem, you will need to justify your answers carefully—unsubstantiated answers will receive little or no credit (except) if the directions for that question specifically say no justification is necessary, such as in the True/False section). Please be sure to write neatly—illegible answers will receive little or no credit. If more space is needed, use the back of the previous page to continue your work. Be sure to make a note of this on the problem page so that the grader knows where to find your answers. You are allowed one 4 by 6 inch file card of notes to use on the examination. No calculators or other aids are allowed.

1. (12 points) True or False. No justification is necessary, simply circle **T** or **F** for each statement.

T **F** (a) If $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly independent set in $\mathbb{R}^n$, then $\{\mathbf{v}_1+\mathbf{v}_2, \mathbf{v}_2+\mathbf{v}_3, \mathbf{v}_1+\mathbf{v}_3\}$ is also a linearly independent set in $\mathbb{R}^n$.

T **F** (b) It is possible to have a 5×3 matrix A such that the dimension of the kernel of A is four.

T **F** (c) If $A^2 + 2A - 5I_3 = 0$ for a 3×3 matrix A, then A is invertible.

T **F** (d) If A and B are $n \times n$ matrices and $\mathbf{x}$ is in the kernel of A, then $\mathbf{x}$ must also be in the kernel of AB.

T **F** (e) Row operations on an $m \times n$ matrix A can change the kernel of A.

T **F** (f) If A and B are $m \times n$ matrices, then

$$\text{rank}(A + B) = \text{rank}(A) + \text{rank}(B).$$

2. (a) (5 points) Let

$$A = \begin{pmatrix} 1 & -9 & 3 \\ -1 & -4 & 2 \\ 2 & -5 & 1 \end{pmatrix} \text{ and } \mathbf{y} = \begin{pmatrix} 5 \\ -8 \\ \alpha \end{pmatrix}$$

For what value(s) of α, if any, will $\mathbf{y}$ be in the image of A.

(b) (5 points)

$$B = \begin{pmatrix} 1 & 5 & -3 \\ -1 & -4 & 1 \\ -2 & -7 & 0 \end{pmatrix} \text{ and } \mathbf{x} = \begin{pmatrix} -14 \\ 4 \\ \beta \end{pmatrix}$$

For what value(s) of β, if any, will $\mathbf{x}$ be in the kernel of B.

3. Let $T_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be the rotation by angle θ in a counterclockwise direction, and let $S_a : \mathbb{R}^2 \to \mathbb{R}^2$ be the shear transformation that fixes the x-axis and maps $(0,1)$ to $(a,1)$. That is, $S_a(x_1, x_2) = (x_1 + ax_2, x_2)$

   (a) (6 points) Find the matrix of the transformation $S_a \circ T_\theta$.

   (b) (6 points) Show that the inverse of $S_a \circ T_\theta$ has the form $T_\phi \circ S_b$, and find ϕ and b in terms of θ and a.

4. (a) (6 points) Find all values of α such that

$$A = \begin{pmatrix} 2 & 1 & -1 \\ 2 & 2 & 1 \\ 2 & 1 & \alpha \end{pmatrix}$$

is invertible.

(b) (6 points) If $\alpha = 0$, find the inverse of A if it exists.

5. Consider a linear system whose augmented matrix is of the form

$$\left(\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 1 & 2 & 4 & 3 \\ 1 & 3 & \alpha & \beta \end{array}\right).$$

(a) (5 points) For what values of α and β will the system have infinitely many solutions?

(b) (5 points) For what values of α and β will the system be inconsistent?

6. Let

$$A = \begin{pmatrix} 1 & 1 & 1 & 2 \\ -1 & 0 & 2 & -3 \\ 2 & 4 & 8 & 5 \end{pmatrix}.$$

(a) (6 points) Find a basis for $\ker(A)$.

(b) (5 points) Find a basis for $\text{image}(A)$.

7. The graph of $y = ax^2 + bx + c$ is a parabola.

(a) (6 points) If the parabola $y = ax^2 + bx + c$ passes through the points $(-1, -1)$, $(1, 1)$, and $(-2, 2)$, find a, b, and c.

(b) (5 points) Find all parabolas $y = ax^2 + bx + c$ that pass through the points $(-1, -1)$ and $(1, 1)$.

8. (6 points) Suppose two 2×2 matrices A and B that both commute with

$$C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix};$$

that is, $AC = CA$ and $BC = CB$.

(a) (6 points) Show that $AB = BA$.

(b) (5 points) Find C^{1003}.

9. Let U and V be two subspaces of $\mathbb{R}^n$. Let $U \cap V$ be the intersection of U and V. That is, it is the set of all vectors in $\mathbb{R}^n$ that are in *both* U and V.

   (a) (6 points) Show that $U \cap V$ is itself a subspace of $\mathbb{R}^n$.

   (b) (5 points) If U and V are both 1-dimensional, what are the possible dimensions of $U \cap V$ and when does each case occur? [*Hint*: Think of what would happen in $\mathbb{R}^2$ and $\mathbb{R}^3$.]