

Math 21b Midterm I—Solutions
Thursday, March 6, 2003

1. (12 points) True or False. No justification is necessary, simply circle **T** or **F** for each statement.

T **F** (a) If $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly independent set in \mathbb{R}^n , then $\{\mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_2 + \mathbf{v}_3, \mathbf{v}_1 + \mathbf{v}_3\}$ is also a linearly independent set in \mathbb{R}^n .

Solution. This statement is true. Suppose that

$$a(\mathbf{v}_1 + \mathbf{v}_2) + b(\mathbf{v}_2 + \mathbf{v}_3) + c(\mathbf{v}_1 + \mathbf{v}_3) = (a+c)\mathbf{v}_1 + (a+b)\mathbf{v}_2 + (b+c)\mathbf{v}_3 = \mathbf{0}.$$

Then

$$\begin{aligned} a + c &= 0 \\ a + b &= 0 \\ b + c &= 0, \end{aligned}$$

since $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly independent set. If we solve this system, then we see that $a = b = c = 0$. Therefore, the vectors $\mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_2 + \mathbf{v}_3, \mathbf{v}_1 + \mathbf{v}_3$ must be linearly independent.

T **F** (b) It is possible to have a 5×3 matrix A such that the dimension of the kernel of A is four.

Solution. This statement is false. Since $\text{rank}(A) + \text{nullity}(A) = 3$, the dimension of the kernel of A must be less than or equal to 3.

T **F** (c) If $A^2 + 2A - 5I_3 = 0$ for a 3×3 matrix A , then A is invertible.

Solution. This statement is true. Solving for I_3 ,

$$I_3 = A \left(\frac{1}{5}A + \frac{2}{5}I_3 \right).$$

Therefore,

$$A^{-1} = \frac{1}{5}A + \frac{2}{5}I_3.$$

T **F** (d) If A and B are $n \times n$ matrices and \mathbf{x} is in the kernel of A , then \mathbf{x} must also be in the kernel of AB .

Solution. This statement is false. Let

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then

$$\mathbf{x} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

is in the kernel of A but is not in the kernel of AB .

T F (e) Row operations on an $m \times n$ matrix A can change the kernel of A .

Solution. This statement is false, since it statement contradicts Gauss-Jordan elimination.

T F (f) If A and B are $m \times n$ matrices, then

$$\text{rank}(A + B) = \text{rank}(A) + \text{rank}(B).$$

Solution. This statement is false. Let

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

2. (a) (5 points) Let

$$A = \begin{pmatrix} 1 & -9 & 3 \\ -1 & -4 & 2 \\ 2 & -5 & 1 \end{pmatrix} \text{ and } \mathbf{y} = \begin{pmatrix} 5 \\ -8 \\ \alpha \end{pmatrix}$$

For what value(s) of α , if any, will \mathbf{y} be in the image of A .

Solution. We must find \mathbf{x} such that $A\mathbf{x} = \mathbf{y}$. That is, we must solve the system whose augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & -9 & 3 & 5 \\ -1 & -4 & 2 & -8 \\ 2 & -5 & 1 & \alpha \end{array} \right).$$

However, this matrix is row equivalent to the matrix

$$\left(\begin{array}{ccc|c} 1 & -9 & 3 & 5 \\ 0 & -13 & 5 & -3 \\ 0 & 0 & 0 & \alpha - 13 \end{array} \right).$$

Therefore, \mathbf{y} is in the image of A if and only if $\alpha = 13$.

(b) (5 points)

$$B = \begin{pmatrix} 1 & 5 & -3 \\ -1 & -4 & 1 \\ -2 & -7 & 0 \end{pmatrix} \text{ and } \mathbf{x} = \begin{pmatrix} -14 \\ 4 \\ \beta \end{pmatrix}$$

For what value(s) of β , if any, will \mathbf{x} be in the kernel of B .

Solution. The vector \mathbf{x} is in the kernel of B if $B\mathbf{x} = \mathbf{0}$. However,

$$\begin{aligned} B\mathbf{x} &= -14 \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} + 4 \begin{pmatrix} 5 \\ -4 \\ -7 \end{pmatrix} + \beta \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 6 - 3\beta \\ -2 + \beta \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \end{aligned}$$

Therefore, $\beta = 2$.

3. Let $T_\theta : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the rotation by angle θ in a counterclockwise direction, and let $S_a : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the shear transformation that fixes the x -axis and maps $(0, 1)$ to $(a, 1)$. That is, $S_a(x_1, x_2) = (x_1 + ax_2, x_2)$

(a) (6 points) Find the matrix of the transformation $S_a \circ T_\theta$.

Solution. The matrices for T_θ and S_a are

$$\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \text{ and } \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix},$$

respectively. Therefore, the matrix for $S_a \circ T_\theta$ is

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} = \begin{pmatrix} \cos \theta + a \sin \theta & -\sin \theta + a \cos \theta \\ \sin \theta & \cos \theta \end{pmatrix}.$$

(b) (6 points) Show that the inverse of $S_a \circ T_\theta$ has the form $T_\phi \circ S_b$, and find ϕ and b in terms of θ and a .

Solution. Since

$$\begin{aligned} \left[\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \right]^{-1} &= \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}^{-1} \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{pmatrix} \begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix}, \end{aligned}$$

we know that $\phi = -\theta$ and $b = -a$.

4. (a) (6 points) Find all values of α such that

$$A = \begin{pmatrix} 2 & 1 & -1 \\ 2 & 2 & 1 \\ 2 & 1 & \alpha \end{pmatrix}$$

is invertible.

Solution. The matrix A is row equivalent to

$$\begin{pmatrix} 2 & 1 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & \alpha + 1 \end{pmatrix}$$

For A to have an inverse, it must be the case that $\text{rank}(A) = 3$. However, $\text{rank}(A) = 3$ if and only if $\alpha \neq -1$.

- (b) (6 points) If $\alpha = 0$, find the inverse of A if it exists.

Solution. The matrix

$$\left(\begin{array}{ccc|ccc} 2 & 1 & -1 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 & 0 & 1 \end{array} \right)$$

is row equivalent to

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/2 & -1/2 & 3/2 \\ 0 & 1 & 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array} \right).$$

Therefore,

$$A^{-1} = \begin{pmatrix} -1/2 & -1/2 & 3/2 \\ 1 & 1 & -2 \\ -1 & 0 & 1 \end{pmatrix}.$$

5. Consider a linear system whose augmented matrix is of the form

$$\left(\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 1 & 2 & 4 & 3 \\ 1 & 3 & \alpha & \beta \end{array} \right).$$

- (a) (5 points) For what values of α and β will the system have infinitely many solutions?

Solution. Since the row echelon form of the system is

$$\left(\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & \alpha - 5 & \beta - 4 \end{array} \right),$$

the system will only have free variables and an infinite number of solutions if $\alpha = 5$ and $\beta = 4$.

- (b) (5 points) For what values of α and β will the system be inconsistent?

Solution. The system will be inconsistent when $\alpha = 5$ and $\beta \neq 4$.

6. Let

$$A = \begin{pmatrix} 1 & 1 & 1 & 2 \\ -1 & 0 & 2 & -3 \\ 2 & 4 & 8 & 5 \end{pmatrix}.$$

- (a) (6 points) Find a basis for $\ker(A)$.

Solution. Since

$$\text{rref}(A) = \begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

we know that the kernel of A is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = t \begin{pmatrix} 2 \\ -3 \\ 1 \\ 0 \end{pmatrix}.$$

Therefore, the vector

$$\begin{pmatrix} 2 \\ -3 \\ 1 \\ 0 \end{pmatrix}$$

forms a basis for the kernel of A .

(b) (5 points) Find a basis for $\text{image}(A)$.

Solution. We can find a basis for the image of A by selecting the pivot columns of A :

$$\left\{ \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix} \right\}.$$

7. The graph of $y = ax^2 + bx + c$ is a parabola.

(a) (6 points) If the parabola $y = ax^2 + bx + c$ passes through the points $(-1, -1)$, $(1, 1)$, and $(-2, 2)$, find a , b , and c .

Solution. If we substitute the points into $y = ax^2 + bx + c$, we find that we get the system

$$\begin{aligned} a - b + c &= -1 \\ a + b + c &= 1 \\ 4a - 2b + c &= 2. \end{aligned}$$

Solving this system, we have $a = 4/3$, $b = 1$, and $c = -4/3$.

(b) (5 points) Find all parabolas $y = ax^2 + bx + c$ that pass through the points $(-1, -1)$ and $(1, 1)$.

Solution. Solving the system

$$\begin{aligned} a - b + c &= -1 \\ a + b + c &= 1, \end{aligned}$$

we find that the parabolas that pass through the points $(-1, -1)$ and $(1, 1)$ must have the form

$$ax^2 + x - a.$$

8. (6 points) Suppose two 2×2 matrices A and B that both commute with

$$C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix};$$

that is, $AC = CA$ and $BC = CB$.

- (a) (6 points) Show that $AB = BA$.

Solution. If

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

then

$$AC = \begin{pmatrix} -b & a \\ -d & c \end{pmatrix} \text{ and } CA = \begin{pmatrix} c & d \\ -a & -b \end{pmatrix}.$$

Since $AC = CA$, we know that $a = d$ and $b = -c$. Therefore,

$$A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

Similarly,

$$B = \begin{pmatrix} x & -y \\ y & x \end{pmatrix}.$$

Since

$$AB = \begin{pmatrix} ax - by & -(ay + bx) \\ ay + bx & ax - by \end{pmatrix} = BA,$$

the matrices commute. We could also say that since A and B are rotation-dilation matrices, $AB = BA$.

- (b) (5 points) Find C^{1003} .

Solution. First notice that $C^4 = I_2$. Then

$$C^{1003} = (C^4)^{250}C^3 = C^3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

9. Let U and V be two subspaces of \mathbb{R}^n . Let $U \cap V$ be the intersection of U and V . That is, it is the set of all vectors in \mathbb{R}^n that are in *both* U and V .

- (a) (6 points) Show that $U \cap V$ is itself a subspace of \mathbb{R}^n .

Solution. We must verify that $U \cap V$ satisfies the definition of a subspace. That is, we must show that $U \cap V$ contains the zero vector and is closed under vector addition and scalar multiplication.

- Since U and V are both subspaces of \mathbb{R}^n , the zero vector is in both U and V . Therefore, $\mathbf{0} \in U \cap V$.

- Now suppose that $\mathbf{x}, \mathbf{y} \in U \cap V$. Then $\mathbf{x}, \mathbf{y} \in U$ and $\mathbf{x}, \mathbf{y} \in V$. Since U and V are both subspaces, $\mathbf{x} + \mathbf{y} \in U$ and $\mathbf{x} + \mathbf{y} \in V$. Thus, $\mathbf{x} + \mathbf{y} \in U \cap V$.
 - If $\mathbf{x} \in U \cap V$, then $\mathbf{x} \in U$ and $\mathbf{x} \in V$. Since both U and V are subspaces, $\alpha\mathbf{x} \in U$ and $\alpha\mathbf{x} \in V$ for any number $\alpha \in \mathbb{R}$. Therefore, $\alpha\mathbf{x} \in U \cap V$.
- (b) (5 points) If U and V are both 1-dimensional, what are the possible dimensions of $U \cap V$ and when does each case occur? [*Hint*: Think of what would happen in \mathbb{R}^2 and \mathbb{R}^3 .]

Solution. If $U = V$, then $\dim U \cap V = 1$. If $U \neq V$, then $\dim U \cap V = 0$.