Math 21b Midterm 2 Solutions - Fall 2001

1. (a) The set W is the set of solutions (x, y, z) to the system $2x - y + 4z = 0$. Thus this space is spanned by $\begin{bmatrix} .5 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$. These are linearly independent (because they are not multiples of each of each other), so they are a basis.

   (b) To change the above basis into an orthonormal basis, apply the Gram-Schmidt process:

$$\begin{bmatrix} .5 \\ 1 \\ 0 \end{bmatrix} \longrightarrow \frac{2}{\sqrt{5}} \begin{bmatrix} .5 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \\ 0 \end{bmatrix}.$$

$$\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} - \left(\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \\ 0 \end{bmatrix} \right) \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 2/5 \\ 4/5 \\ 0 \end{bmatrix} = \begin{bmatrix} -1.6 \\ 0.8 \\ 1 \end{bmatrix},$$

$$\text{so} \quad \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \longrightarrow \sqrt{\frac{5}{21}} \begin{bmatrix} -1.6 \\ 0.8 \\ 1 \end{bmatrix} = \begin{bmatrix} -8/\sqrt{105} \\ 4/\sqrt{105} \\ \sqrt{5}/\sqrt{21} \end{bmatrix}.$$

   So an orthonormal basis for this space is $\left\{ \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \\ 0 \end{bmatrix}, \begin{bmatrix} -8/\sqrt{105} \\ 4/\sqrt{105} \\ \sqrt{5}/\sqrt{21} \end{bmatrix} \right\}$.

   (c) The shortest vector from a point to a plane is along the normal vector to the plane through that point. Therefore this vector is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$ for some value t. So when is this vector in the set W? $2(1 + 2t) - (1 - t) + 4(1 + 4t) = 0$ is satisfied for $t = -5/21$. Thus the shortest distance is the norm of $\begin{bmatrix} -10/21 \\ 5/21 \\ -20/21 \end{bmatrix}$, which is $5/\sqrt{21}$.

2. (a) $\det(\lambda I - A) = \det \begin{bmatrix} \lambda - 1 & -4 & -5 \\ 0 & \lambda - 2 & -6 \\ 0 & 0 & \lambda - 3 \end{bmatrix} = (\lambda - 1)(\lambda - 2)(\lambda - 3)$.

   So $f_A(\lambda) = (\lambda - 1)(\lambda - 2)(\lambda - 3)$.

   (b) A is a 3 x 3 matrix with 3 distinct real eigenvalues, so A is indeed diagonalizable.

(c) I is *definitely* diagonalizable since it has three distinct real eigenvalues. II is *possibly* diagonalizable since the eigenvalue 1 has geometric multiplicity at most 2 - if its multiplicity is 2 than the matrix is diagonalizable because you can find an eigenbasis. III is *definitely not* diagonalizable since there is only one real eigenvalue, and its algebraic multiplicity is 1 so there cannot possibly be an eigenbasis.

3. (a) $A\vec{v}_1 = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4-2 \\ 2+2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} = 2\vec{v}_1$. So $\vec{v}_1$ is an eigenvector with eigenvalue 2.

(b) $A\vec{v}_2 = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4-1 \\ 2+1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3\vec{v}_2$. So $\vec{v}_2$ is an eigenvector with eigenvalue 3.

(c) Recall that for such a system, $\vec{x}(t) = A^t\vec{x}_0$. Using the eigenbasis we found in the previous parts of this problem, we can diagonalize A:

$$S = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}, \quad D = S^{-1}AS = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}.$$

Thus

$$\vec{x}(t) = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2^t & 0 \\ 0 & 3^t \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} -(2^{t+3}) + 11 \cdot 3^t \\ 2^{t+4} - 11 \cdot 3^t \end{bmatrix}.$$

4. (a) $\sqrt{\cos^2\theta + \sin^2\theta} = 1$, and $(\cos\theta)(-\sin\theta) + (\sin\theta)(\cos\theta) = 0$, so any 2 x 2 rotation matrix is an orthogonal matrix.

(b) First notice that any 2 x 2 rotation matrix has determinant 1. Therefore the inverse of a rotation matrix as given in the definition has the form:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^T.$$

(c) The matrix M does *not* have to be a rotation matrix. For example, consider the matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. This matrix is clearly orthogonal, and since $1 \neq -1$, this matrix cannot be a rotation matrix.

5. (a) $\begin{vmatrix} c = 27 \\ a + b + c = 0 \\ 4a + 2b + c = 0 \\ 9a + 3b + c = 0 \end{vmatrix}.$

The bottom three equations in this system are linearly independent, so the only solutions is $(0, 0, 0)$. But this is inconsistent with the first equation $c = 27$.

(b) $$\begin{bmatrix} 0 & 1 & 4 & 9 \\ 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \end{bmatrix} \vec{x} = \begin{bmatrix} 0 & 1 & 4 & 9 \\ 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 27 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 98 & 36 & 14 \\ 36 & 14 & 6 \\ 14 & 6 & 4 \end{bmatrix} \vec{x} = \begin{bmatrix} 0 \\ 0 \\ 27 \end{bmatrix}.$$

(c) Solving this system by row reducing the matrix, we get the polynomial $f(x) = 6.75x^2 - 28.35x + 25.65$.

6. (a) We know that the diagonalization of A ($S^{-1}AS$) has the same eivenvalues as A, and this matrix is diagonal so its eigenvalues are just the diagonal entries. Therefore, the eigenvalues of A are $1, -7$, and 3 (multiplicity 2).

   (b) We know that the matrix A is diagonalizable, so it has an eigenbasis. Thus the geometric multiplicities of the eigenvalues must equal the algebraic multiplicities.

Eigenvalue of A	Algebraic Multiplicity	Geometric Multiplicity
1	1	1
-7	1	1
3	2	2

   (c) $\det(S^{-1}AS) = \det(S^{-1})\det(A)\det(S) = (\det(S))^{-1}\det(A)\det(S) = \det(A)$.

   (d) Since it is easy to find $\det(S^{-1}AS)$ - just multiply the diagonal entries, the previous question tells us that $\det(A) = (1)(-7)(3)^2 = -63$. Thus, since $\det(A) \neq 0$, A is invertible.

7. We know $A\vec{u} = \lambda_1\vec{u}$ and $A\vec{v} = \lambda_2\vec{v}$.

$$\lambda_1(\vec{u}\cdot\vec{v}) = \lambda_1\vec{u}^T\vec{v} = (A\vec{u})^T\vec{v} = \vec{u}^T A^T\vec{v} = \vec{u}^T A\vec{v} = \lambda_2(\vec{u}\cdot\vec{v}).$$

Thus, since $\lambda_1 \neq \lambda_2$, it must be that $\vec{u}\cdot\vec{v} = 0$, so $\vec{u}$ and $\vec{v}$ are orthogonal.