Fall 99 Midterm 2 Solutions

① (a) A is 3×3, so $f_A(\lambda)$ should be a cubic polynomial
$tr(A)=3$, so the coefficient of λ^2 should be -3

(b) $\det(\lambda I_5 - A) = \det \begin{bmatrix} \lambda-5 & 0 & 0 & 0 & 0 \\ * & \lambda-5 & 0 & 0 & 0 \\ * & * & * & 0 & 0 \\ * & * & * & * & 0 \\ * & * & * & * & * \end{bmatrix}$

Each pattern must have a nonzero term in the 1^{st} row and the 2^{nd} row; these can't both be in the 1^{st} column, so there must be 2 factors of $\lambda-5$ in every pattern.

(c) To get B from A:

1) Multiply col. 1 by 100 (det $\times 100$)
2) Add col. 3 to col. 1 (det unchanged)
3) Swap col 3 & col 1 (det $\times -1$)

So $\det B = \det A \times -100$

(d) Volume of $P = \left| \det \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 1 \end{bmatrix} \right| = 9$

T expands the volume by a factor of 9, so Volume of $T(P) = 81$.

(2) i. A matrix of size $n \times n$ is orthogonal if its corresponding transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$ preserves the lengths of vectors.

ii. Since A is orthogonal, the vectors $\vec{v}, A\vec{v}, A^2\vec{v}, \ldots, A^{1999}\vec{v}$

all have the same length, so A^{1999} is orthogonal.

iii. If A is orthogonal, $\det A = \pm 1 \neq 0$, so A is invertible $(A^{-1} = A^T)$

(3) i. Eigenvalues are 0 (if vector is in orthogonal complement) and 1 (if vector is in P)

Since $\dim P = 2$, $\dim P^{\perp} = 3 - 2 = 1$. These are the dimensions of the eigenspaces for 1 and 0 respectively.

ii. Use Gram-Schmidt to find an orthonormal basis for P

$$\vec{w}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$$

$$\vec{v}_2 - (\vec{w}_1 \cdot \vec{v}_2)\vec{w}_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} - \left(\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} \right) \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} - \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \\ -2 \end{bmatrix} \xrightarrow{\text{normalized}} \begin{bmatrix} \sqrt{2}/6 \\ -\sqrt{2}/6 \\ -2\sqrt{2}/3 \end{bmatrix} = \vec{w}_2$$

To find a vector in $P^{\perp}$, find $\ker \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -2 \end{bmatrix} = \operatorname{Span}\left(\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \right)$

$$\vec{w}_3 = \begin{bmatrix} 0 \\ 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$$

iii) Algebraic multiplicity $\geq$ Geometric multiplicity

Since the algebraic multiplicities add to 3,

algebraic multiplicity of $1 = 2$
" " " $0 = 1$

Characteristic polynomial $= (\lambda-1)(\lambda-1)(\lambda-0)$

$= \lambda^3 - 2\lambda^2 + \lambda$

T is not invertible, since det(T) = -product of eigenvalues = 0.

(4) $D(k+1) = .6\,D(k) + .2\,T(k)$
$T(k+1) = .3\,D(k) + .5\,T(k)$

i) $\begin{bmatrix} D(k+1) \\ T(k+1) \end{bmatrix} = \underbrace{\begin{bmatrix} .6 & .2 \\ .3 & .5 \end{bmatrix}}_{A} \begin{bmatrix} D(k) \\ T(k) \end{bmatrix} \qquad \underbrace{\begin{bmatrix} D(0) \\ T(0) \end{bmatrix} = \begin{bmatrix} 1000 \\ 0 \end{bmatrix}}_{x_0}$

ii) $f_A(\lambda) = \det\begin{pmatrix} \lambda-.6 & -.2 \\ -.3 & \lambda-.5 \end{pmatrix} = (\lambda-.6)(\lambda-.5) - .06$

$= \lambda^2 - 1.1\lambda + .24$

$= (\lambda - .8)(\lambda - .3)$

Eigenvalues are .8, .3

$E_{.8} = \ker\begin{pmatrix} .2 & -.2 \\ -.3 & .3 \end{pmatrix} = \operatorname{Span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$

$E_{.3} = \ker\begin{pmatrix} -.3 & -.2 \\ -.3 & -.2 \end{pmatrix} = \operatorname{Span}\left(\begin{bmatrix} 2 \\ -3 \end{bmatrix}\right)$

Eigenvectors

④ $\begin{bmatrix} D(k) \\ T(k) \end{bmatrix} = A^k x_0$

$$x_0 = \begin{bmatrix} 1000 \\ 0 \end{bmatrix} = 600 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 200 \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

$$A^k x_0 = (.8)^k \, 600 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + (.3)^k \, 200 \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

$$D(k) = 600\,(.8)^k + 400\,(.3)^k$$
$$T(k) = 600\,(.8)^k - 600\,(.3)^k$$

As $k \to \infty$, $D(k), T(k) \to 0$.