SPRING 95 2nd EXAM SOLUTIONS

① $\det(A) = \boxed{-12}$

② LEAST SQUARES SOLUTION OF $A\vec{x} = \vec{b}$ is given by the normal equation $A^T A \vec{x} = A^T \vec{b}$

In this case $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 1 \end{bmatrix}$ $\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

So $A^T A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 4 & 6 \end{bmatrix}$ $A^T\vec{b} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ $\Rightarrow \begin{bmatrix} 3 & 4 \\ 4 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \Rightarrow \boxed{\begin{matrix} x_1 = 0 \\ x_2 = 1/2 \end{matrix}}$

③ $\lambda I - A = \begin{bmatrix} \lambda & 0 & -3 \\ 0 & \lambda-2 & 0 \\ -3 & 0 & \lambda \end{bmatrix}$ $f_A(\lambda) = \det(\lambda I - A) = \lambda^3 - 2\lambda^2 - 9\lambda + 18$

$= (\lambda-2)(\lambda-3)(\lambda+3)$

$\Rightarrow$ real, distinct eigenvalues $\{3, 2, -3\}$

$\lambda_1 = 3 \Rightarrow \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ $\lambda_2 = 2 \Rightarrow \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $\lambda_3 = -3 \Rightarrow \vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ eigenvectors

If $P = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$, then $P^{-1}AP = D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$

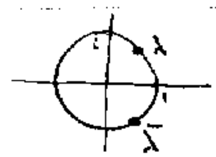
$\Rightarrow A = PDP^{-1}$ $A^{100} = PD^{100}P^{-1}$ CALCULATE $P^{-1} = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & -1/2 \end{bmatrix}$

$A^{100}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = PD^{100}P^{-1}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix}\begin{bmatrix} 3^{100} & 0 & 0 \\ 0 & 2^{100} & 0 \\ 0 & 0 & (-3)^{100} \end{bmatrix}\begin{bmatrix} 1/2 \\ 0 \\ 1/2 \end{bmatrix}$

$= \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix}\begin{bmatrix} \frac{1}{2}\cdot 3^{100} \\ 0 \\ \frac{1}{2}\cdot 3^{100} \end{bmatrix} = \frac{1}{2}(3^{100})[\vec{v}_1 + \vec{v}_3] = \frac{1}{2}(3^{100})\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3^{100} \\ 0 \\ 0 \end{bmatrix}$

④ $\lambda I - M = \begin{bmatrix} \lambda-2 & 3 \\ -1 & \lambda+1 \end{bmatrix}$ $f_M(\lambda) = \det(\lambda I - M) = \lambda^2 - \lambda + 1 = 0$

$\Rightarrow \lambda = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$



$\lambda = \frac{1}{2} + i\frac{\sqrt{3}}{2}$, $\bar{\lambda} = \frac{1}{2} - i\frac{\sqrt{3}}{2}$

$\Rightarrow \{e^{i\pi/3}, e^{-i\pi/3}\}$

$\lambda = \frac{1}{2} + i\frac{\sqrt{3}}{2} \Rightarrow \left[\begin{array}{cc|c} -\frac{3}{2} + i\frac{\sqrt{3}}{2} & 3 & 0 \\ -1 & \frac{3}{2} + i\frac{\sqrt{3}}{2} & 0 \end{array}\right] \Rightarrow \vec{w} = \begin{bmatrix} 3 + i\sqrt{3} \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} + i\begin{bmatrix} \sqrt{3} \\ 0 \end{bmatrix} = \vec{u} + i\vec{v}$

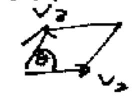
$\mathcal{B} = \{\vec{v}, \vec{u}\}$ basis $P = \begin{bmatrix} \sqrt{3} & 3 \\ 0 & 2 \end{bmatrix} \Rightarrow P^{-1}MP = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix} = R_{\pi/3}$ (ROTATION through $60°$)

$M = PR_{\pi/3}P^{-1}$ $M^t = PR_{\pi t/3}P^{-1}$ $\vec{x}(t) = M^t\vec{x}_0$.

Except for coord. change, system is just discrete Rotation through $60° \Rightarrow$ Periodic

$K = 6$ is smallest K for which $M^K = I_2$ ($\frac{6\pi}{3} = 2\pi \rightarrow$ full rotation)

⑤ DATA gives $\|\vec{v}_1\| = 2$, $\|\vec{v}_2\| = \sqrt{2}$, $\|\vec{v}_3\| = \sqrt{5}$ and $\vec{v}_1 \perp \vec{v}_2$, $\vec{v}_1 \perp \vec{v}_3$.



$(\text{Area}(\vec{v}_2, \vec{v}_3))^2 = \|\vec{v}_2\|^2\|\vec{v}_3\|^2\sin^2\theta = \|V_2\|^2\|\vec{V}_3\|^2(1 - \cos^2\theta)$

$= \|\vec{v}_2\|^2\|\vec{v}_3\|^2 - (\vec{v}_2\cdot\vec{v}_3)^2 = 2\cdot 5 - 1^2 = 10 - 1 = 9 \Rightarrow \text{Area} = 3$

Finally, VOLUME $= (\text{area})(\perp\text{height}) = 3\|\vec{v}_1\| = 3\cdot 2 = \boxed{6}$