Mathematics 21b - Second Exam - Fall 2000

This exam was 90 minutes long. No calculators were allowed.

(1) (30 points) True or False? (Circle one and give just a few words of explanation.)

(a) If **A** is an $n \times n$ matrix, then det(2**A**) = 2 (det **A**). True False

(b) Let **A** be a 100×100 matrix with every entry equal to 1. Then det **A** = 1. True False

(c) Let **A** be a square matrix with exactly one entry 1 in each row and in each column, the other entries being zero. Then **A** is an orthogonal matrix. True False

(d) If **v** is an eigenvector of both **A** and **B**, then **v** is an eigenvector of **A** + **B**. True False

(e) If **v** is an eigenvector of both **A** and **B**, then **v** is an eigenvector of **AB**. True False

(f) A projection is an orthogonal transformation. True False

(g) If a 7×7 matrix **A** has seven different eigenvalues, then **A** is diagonalizable. True False

(h) If a 7×7 matrix **A** has six eigenvalues, then **A** is not diagonalizable. True False

(i) If **A** is an $n \times n$ matrix, then **A** and $\mathbf{A}^T$ have the same eigenvalues. True False

(j) If **A** is an $n \times n$ matrix, then **A** and $\mathbf{A}^T$ have the same eigenvectors. True False

(k) If **v** is a unit (column) vector in $\mathbf{R}^3$, then the matrix $\mathbf{v}\mathbf{v}^T$ is diagonalizable. True False

(l) If **A** is a 4×4 matrix with $\mathbf{A}^2 = \mathbf{A}$, then the only possible eigenvalues of **A** are 0 and 1. True False

(2) (20 points) Consider the matrix

$$\mathbf{A} = \begin{bmatrix} -1 & 0 & 1 & 0 & 3 \\ 1 & 0 & 3 & 4 & 1 \\ 1 & 0 & 1 & 2 & -1 \end{bmatrix}.$$

(a) Find a basis for ker **A**, the kernel of **A**.

(b) Find a basis for im **A**, the image of **A.**

(c) Using the Gram-Schmidt process on your answer to (b), find an orthonormal basis for im **A**.

(d) Find the matrix for orthogonal projection onto the subspace im **A**.

(e) Find the orthogonal projection of the standard basis vector $\mathbf{j} = \mathbf{e}_2$ in the subspace im **A**.

(3) (15 points) Let **B** be the matrix $\mathbf{B} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$.

a) Find det(**B**).

b) Find all eigenvalues (real and/or complex) of **B** and their **algebraic** multiplicities.

c) For each **real** eigenvalue of **B**, find the eigenspace corresponding to it and its geometric multiplicity. (Clearly indicate which eigenspaces and multiplicities belong to which eigenvalues.)

(4) (20 points)

(a) Find a real 2×2 matrix **A** such that $\vec{x}(t) = \begin{bmatrix} 2^t \\ 1-2^t \end{bmatrix}$ is a trajectory of the dynamical system $\vec{x}(t+1) = \mathbf{A}\,\vec{x}(t)$.

(b) What are the eigenvalues and eigenvectors of **A**?

(c) Calculate $\mathbf{A}^5$.

(5) (15 points) Let T: $\mathbf{R}^3 \to \mathbf{R}^3$ be orthogonal projection onto the z-axis. Find the matrix M of this linear transformation, <u>with respect to the orthonormal basis:</u>

$$\mathbf{v}_1 = \tfrac{1}{7}\begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix},\ \mathbf{v}_2 = \tfrac{1}{7}\begin{bmatrix} 6 \\ 2 \\ -3 \end{bmatrix},\ \mathbf{v}_3 = \tfrac{1}{7}\begin{bmatrix} -3 \\ 6 \\ -2 \end{bmatrix}.$$