

MATH 21b

FALL 2001

EXAMINATION 2

Tuesday, 4 December, 2001.

Name: _____

Teaching Fellow (PLEASE CIRCLE)

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Instructions:

1. **Do not open this test until told to do so.**
2. Please do not detach any pages from this exam.
3. You may use your calculator and one (1) page of notes not exceeding 8.5 by 11 inches in size.
4. You may use the backs of test sheets for scratch paper, or to continue your working on problems. If you write on the backs of the test sheets, **please label your working very clearly.**
5. **SHOW ALL YOUR WORK.**
6. Exam proctors are not permitted to answer questions regarding the content of the test.
7. Many of the questions have precisely worded instructions. **Be sure to read all instructions carefully**, and do all that is asked.
8. Please try your best - try to relax and show us what you can do!

Problem	Total	Score
1	13	
2	13	
3	13	
4	16	
5	16	
6	19	
7	10	
Total	100	

1. (13 points total)

The set of vectors: $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : 2x - y + 4z = 0 \right\}$ is a subspace of \mathbf{R}^3 . (There is no need for you to demonstrate this fact.)

(a) (4 points) Find a basis for W .

(b) (4 points) Construct an orthonormal basis for W .

(c) (5 points) Calculate the shortest distance from the point $(1, 1, 1)$ to W .

2. (13 points total)

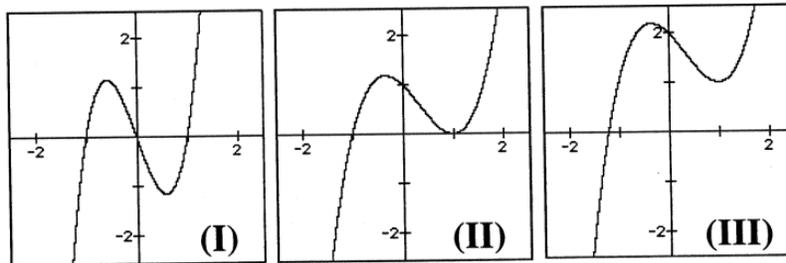
In this problem, the matrix A will always be:

$$A = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{bmatrix}$$

(a) (3 points) Find the characteristic polynomial of the matrix A .

(b) (4 points) Is the matrix A diagonalizable or not? Explain how you can tell.

(c) (6 points) The graphs shown below are the graphs of the characteristic polynomials of three by three matrices. For each, decide if the matrix is definitely diagonalizable, possibly diagonalizable or definitely not diagonalizable.



3. (13 points total)

In this problem the matrix A will always refer to the 2 by 2 matrix:

$$A = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}.$$

(a) (3 points) Show that the vector $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector of A and find the corresponding eigenvalue.

(b) (3 points) Show that the vector $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of A and find the corresponding eigenvalue.

(c) (7 points) Suppose that the matrix A is the matrix for a *discrete* dynamical system:

$$\vec{x}(t) = \begin{bmatrix} c(t) \\ r(t) \end{bmatrix} \quad \vec{x}(t+1) = A \cdot \vec{x}(t) \quad \vec{x}(0) = \begin{bmatrix} 3 \\ 5 \end{bmatrix}.$$

Find an explicit formula for the solution vector $\vec{x}(t)$.

4. (16 points total)

In this question, you will always be dealing with two by two matrices. You are expected to use the two definitions given below to answer this question.

Definition 1: A 2 by 2 *rotation matrix* is a matrix of the form:

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

Definition 2: An *orthogonal matrix* is a square matrix whose columns are orthonormal vectors.

(a) (4 points) Show that a 2 by 2 rotation matrix is an orthogonal matrix.

(b) (6 points) Show that the inverse of a rotation matrix is equal to the transpose of the rotation matrix.

Note: $\sin(-\theta) = -\sin(\theta)$ and $\cos(-\theta) = \cos(\theta)$.

(c) (6 points) Suppose that M is an orthogonal, 2 by 2 matrix that has real number entries. Does M have to be a rotation matrix? If you believe that M must be a rotation matrix, explain why. If you do not believe that M has to be a rotation matrix, give an example.

5. (16 points total)

In this problem you will fit a quadratic polynomial to the data points:

$$(0, 27), (1, 0), (2, 0), (3, 0)$$

using the technique of Least Squares.

- (a) **(4 points)** A *quadratic* polynomial is a function of the form: $f(x) = a \cdot x^2 + b \cdot x + c$. Use the data points given above to set up a system of linear equations. Briefly explain how you can tell that the system of equations is **inconsistent**.

- (b) **(6 points)** The least-squares solution of a system $A\vec{x} = \vec{b}$ is the solution of the corresponding *normal equation*:

$$A^T A \vec{x} = A^T \vec{b}.$$

Set up the normal equation for the system of linear equations that you formulated in Part (a).

- (c) **(6 points)** Find the equation of the quadratic polynomial that best fits the data points given above.

6. (19 points total)

In this problem, the matrix A will always be the matrix:

$$A = \begin{bmatrix} -1 & 20 & -34 & -16 \\ \frac{-23}{2} & \frac{267}{4} & \frac{-441}{4} & \frac{-209}{4} \\ -7 & \frac{75}{2} & \frac{-123}{2} & \frac{-61}{2} \\ \frac{-3}{2} & \frac{35}{4} & \frac{-61}{4} & \frac{-17}{4} \end{bmatrix}$$

It is possible to show that there is an invertible 4 by 4 matrix S such that:

$$S^{-1}AS = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

Please note that it is **not necessary** for you to calculate the matrix S , although if you believe that this will help you solve the problem you are welcome to do so.

- (a) **(6 points)** What are the eigenvalues of the matrix A ?
- (b) **(6 points)** Complete the table given below. (Note: The table has deliberately been constructed with more spaces than you actually need, so don't worry if you have space left over.)

Eigenvalue of A	Algebraic Multiplicity	Geometric Multiplicity

- (c) **(4 points)** Show that: $\det(A) = \det(S^{-1}AS)$.
- (d) **(3 points)** What is the determinant of A ? Is A an invertible matrix? Explain how you know.

7. (10 points total)

Let A be a symmetric n by n matrix. That is, $A = A^T$. Suppose that λ_1 and λ_2 are distinct eigenvalues of A . That is, $\lambda_1 \neq \lambda_2$. Suppose that \vec{u} is an eigenvector of A with eigenvalue λ_1 . Suppose that \vec{v} is an eigenvector of A with eigenvalue λ_2 . Show that the vectors \vec{u} and \vec{v} are orthogonal.

Hint: Recall that $\vec{u} \cdot \vec{v} = \vec{u}^T \vec{v}$.