

Last Name: \_\_\_\_\_

First Name: \_\_\_\_\_

## Mathematics 21b

Second Midterm Examination  
November 22, 1999

Your Section (circle one):

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MWF 10        MWF 11

| Question | Points | Score |
|----------|--------|-------|
| 1        | 12     |       |
| 2        | 8      |       |
| 3        | 10     |       |
| 4        | 10     |       |
| Total    | 40     |       |

No calculators are allowed.

1. Each of the following requires only a sentence or two of argument or explanation: lengthy computations and difficult proofs are not needed. 3 points each.

(a) Let  $A$  be the matrix

$$\begin{bmatrix} 3 & 1 & 4 \\ 1 & 5 & 9 \\ -2 & -6 & -5 \end{bmatrix}$$

Show that the characteristic polynomial  $f_A(\lambda)$  is NOT equal to either  $\lambda^4 - 3\lambda^3 + 36\lambda^2 - 90\lambda$  or  $\lambda^3 + 3\lambda^2 + 36\lambda - 90$ .

(b) Show that, if  $A$  is any matrix of the form

$$\begin{bmatrix} 5 & 0 & 0 & 0 & 0 \\ * & 5 & 0 & 0 & 0 \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix}$$

where each  $*$  is a (possibly different) arbitrary number, then 5 is an eigenvalue of  $A$  of algebraic multiplicity at least 2.

(c) Let  $A$  and  $B$  be the matrices

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 2 \\ 1 & 9 & 9 & 9 \\ 2 & 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 2 & 103 & 4 \\ 9 & 4 & 109 & 2 \\ 9 & 9 & 109 & 9 \\ 0 & 0 & 200 & 1 \end{bmatrix}.$$

Given that  $\det(A) = 186$ , show that  $\det(B) = -18600$ .

(d) Let  $\mathbf{P}$  be the parallelepiped in  $\mathbb{R}^3$  defined by the vectors

$$\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}.$$

What is the volume of  $\mathbf{P}$ ? What is the volume of  $T(\mathbf{P})$ , where  $T$  is the linear transformation with matrix

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 1 \end{bmatrix} ?$$

2. i. (3 points) Define what it means for a matrix  $A$  to be **orthogonal**.

ii. (3 points) Show that if  $A$  is orthogonal, so is  $A^{1999}$ .

iii. (2 points) Is every orthogonal matrix invertible? Why?

3. Let  $P$  be the plane in  $\mathbb{R}^3$  spanned by the column vectors

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}.$$

Let  $T$  be the orthogonal projection to  $P$ , i.e.  $T = \text{proj}_P$ . Note: You do not need to find the matrix for  $T$  to solve any of the following.

i) (3 points) Determine all the eigenvalues of  $T$ . ~~For each eigenvalue, describe its eigenspace, and determine its geometric multiplicity.~~

*For each eigenvalue, describe its eigenspace, and determine the eigenvalue's geometric multiplicity.*

ii) (3 points) Find an *orthonormal* basis for  $\mathbb{R}^3$  consisting of eigenvectors for  $T$ .

iii) (4 points) Find the algebraic multiplicity of each eigenvalue of  $T$ . What is the characteristic polynomial of  $T$ ? Is  $T$  invertible? Why or why not?

4. It is widely known that the Starship Enterprise is powered by Dilithium. Less well known is that of each 10 tons of Dilithium, only 6 remain at the end of a year; of the remaining 4 tons, one is converted to energy to run the warp engines etc., but the other 3 transmute to 3 tons of Trilithium. Trilithium is not stable either: in a year, of each 10 tons only 5 remain, with 3 tons converted to energy and the other 2 tons decaying back to Dilithium.

At Stardate 2000, the Starship's reactor is loaded with 1000 tons of Dilithium.

i) (2 points) Find a matrix  $A$  and an initial vector  $x(0)$  that encode this discrete linear dynamical system, i.e. such that at Stardate  $2000+k$ , the Dilithium and Trilithium remaining in the reactor are the entries of the vector  $x(k) = A^k x_0$ .

ii) (3 points) Find the characteristic polynomial and eigenvalues of  $A$ , and compute a basis of eigenvectors.

iii) (3 points) Compute a formula for the amounts of Dilithium and Trilithium remaining at Stardate  $2000 + k$  as functions of  $k$ . What happens to these for large  $k$ ?

iv) (2 points) Many years and Star Trek episodes later, it is found that only one ton of Dilithium is left in the reactor. Approximately how much Trilithium is there at the same time?