Last Name:______________________________

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Mathematics 21b

Second Exam Solutions
April 15, 2002

Your Section (circle one):

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MWF 10	MWF 11	MWF 12	TuTh 10	TuTh 11:30

Question	Points	Score
1	12	
2	8	
3	7	
4	7	
5	7	
6	9	
Total	50	

The exam will last 90 minutes.
No calculators are allowed.
Justify your answers carefully (except in Questions 1 and 2).
Write your final answers in the spaces provided.

(1) True or False (no explanation is necessary).

T **(F)** **:** The transpose of A^TA is AA^T.

Since $(AB)^T = B^TA^T$, we have $(A^TA)^T = A^T(A^T)^T = A^TA$.

(T) **F** **:** If A is a 7×7 matrix and $\det A = \det(-A)$ then A is not invertible.

If A is an $n \times n$ matrix and n is odd, then $\det(-A) = \det(-I_n)\det(A) = -\det(A)$. But if $\det A = \det(-A)$, then $\det A = -\det A \Rightarrow \det A = 0 \Rightarrow A$ is not invertible (Note that if A were an even-dimensional square matrix this statement would not necessarily be true).

T **(F)** **:** There are 5×5 invertible matrices A and B with $AB + BA = 0$.

Suppose that there are 5×5 invertible matrices A and B such that this is true. Rearranging terms and multiplying on the left by A^{-1} gives

$$AB = -BA \Rightarrow A^{-1}AB = A^{-1}(-B)A \Rightarrow B = A^{-1}(-B)A.$$

In other words, B is similar to $-B$.

Now consider the determinant of B and of $A^{-1}(-B)A$. Since A is invertible, we know that $\det(A^-1) = \frac{1}{\det A}$, so that $\det B = \det(A^{-1}(-B)A) = \det(A^{-1})\det(-B)\det A = \det(-B)\frac{1}{\det A}\det A = \det(-B)$. As in the previous question, however, we know that since B is a 5×5 matrix and $\det B = \det(-B)$, then B is not invertible. We supposed that B was invertible, however, so the statement must be false.

(T) **F** **:** For any $n \times n$ real matrix A, all real eigenvalues of the matrix AA^T are nonnegative.

Note that for an $n \times n$ real matrix A, AA^T is symmetric. Consider an eigenvector $\vec{v}$ with associated real eigenvalue λ: $AA^T\vec{v} = \lambda\vec{v}$. Dotting $\vec{v}$ into both sides, we see that

$$\begin{aligned}
\vec{v} \cdot (AA^T\vec{v}) &= \vec{v}^TAA^T\vec{v} \\
&= (A^T\vec{v})^T(A^T\vec{v}) \\
&= \|A^T\vec{v}\|^2 \\
&= \lambda\vec{v}^T\vec{v} \\
&= \lambda\|\vec{v}\|^2
\end{aligned}$$

Since the norm is always greater than or equal to zero, and since $\vec{v}$ is an eigenvector we may assume that $\|\vec{v}\| \neq 0$, we see that

$$\lambda = \frac{\|A^T\vec{v}\|}{\|\vec{v}\|} \geq 0$$

for all real eigenvalues λ.

(T) F : If A is an $n \times n$ matrix with $A^{10} = 0$, then 0 is an eigenvalue of A.

Let λ be an eigenvalue of A with associated eigenvector $\vec{v}$. Then $A^{10}\vec{v} = \lambda^{10}\vec{v} = 0$, since $A^{10} = 0$. Therefore $\lambda = 0$. (From this we see in fact that *every* eigenvalue of A is 0.)

T (F) : There is a subspace $V \subset \mathbf{R}^3$ such that $\operatorname{Tr}\operatorname{proj}_V = 4$.

Let $V \subset \mathbf{R}^3$ be a subspace with orthonormal basis $\vec{v}_1, \ldots, \vec{v}_i$; since V is a subspace we know that i can be at most 3. The projection of a vector $\vec{x}$ onto V is then given by AA^T, where

$$A = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_i \\ | & & | \end{bmatrix}.$$

AA^T is then at most a 3×3 matrix. Its trace is given by $\text{Tr}AA^T = \sum_{n=1}^{i \leq 3} v_{1_n}^2 + \cdots + v_{i_n}^2$, where $v_k = \begin{bmatrix} v_{k_1} \\ \vdots \\ v_{k_i} \end{bmatrix}$. Rearranging these terms gives $\text{Trproj}_V = \sum_{n=1}^{i} \|\vec{v}_n\|^2$; since the $\vec{v}_i$ are orthonormal we know that $\|\vec{v}_i\|^2 = 1$ for all i. Therefore $\text{Trproj}_V = \sum_{n=1}^{i} 1 = i \leq 3$.

Alternatively, the matrix of the projection onto V has eigenvalues 1 with multiplicity equal to the dmension of V and 0 with multiplicity equal to $3 - \text{dim}V$. The trace is then the sum of the eigenvalues, so we see again that $\text{Trproj}_V = \text{dim}V \leq 3$.

(2) Consider the discrete dynamical system $\vec{x}(t+1) = A\vec{x}(t)$ with

$$A = \begin{bmatrix} -1 & -1 \\ 1+2k & k+1 \end{bmatrix}.$$

In each of the following problems find the value of k from the list below such that the dynamical system defined by A has trajectories which exhibit the indicated behavior. (No explanation is necessary.)

Possible k:

$$k = 1/2,\ 1,\ 2,\ 6$$

(a) Spiral out to infinity.
$k = 2$

We first find the eigenvalues of A:

$$\det(A - \lambda I_2) = (-1-\lambda)(k+1-\lambda) - (-1)(1+2k) = \lambda^2 - k\lambda + k = 0$$

$$\Rightarrow \lambda = \frac{k \pm \sqrt{k^2 - 4k}}{2}$$

For the trajectories to spiral out to infinity the eigenvalues must be complex and have modulus greater than 1. This is satisfied when $k = 2$ (in this case $\lambda_{1,2} = 1 \pm i$).

(b) Moves out to infinity without circling $\vec{0}$.
$k = 6$

To move out to infinity without spiraling, the eigenvalues must be real and have absolute value greater than 1. This is satisfied when $k = 6$ ($\Rightarrow \lambda_{1,2} = 3 \pm \sqrt{3}$).

(c) Spiral into $\vec{0}$.
$k = \frac{1}{2}$

To spiral into the origin the eigenvalues must be complex and have modulus less than 1. This is satisfied when $k = \frac{1}{2}$ ($\Rightarrow \lambda_{1,2} = \frac{1}{4} \pm \frac{\sqrt{7}}{4} i$).

(d) Remain on ellipses.
$k = 1$

Finally, to remain on ellipses the eigenvalues must be complex and have modulus exactly equal to 1. This occurs when $k = 1$ ($\Rightarrow \lambda_{1,2} = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$).

Note: another way to approach this problem is to recall that if A has complex eigenvalues then the orbits are inward spirals, ellipses, or outward spirals depending on whether $\det A < 1, \det A = 1$, or $\det A > 1$, respectively.

(3) (a) Find an orthonormal basis for the kernel of

$$A = \begin{bmatrix} 2 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -1 & -4 \end{bmatrix}.$$

Row-reducing A, we find that

$$\begin{bmatrix} 2 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -1 & -4 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -1 & -4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & -1 & -4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & -1 & 2 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & -4 \end{bmatrix}$$

So we see that $\ker A$ is spanned by $\vec{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \\ 2 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 4 \\ 0 \\ 1 \end{bmatrix}$. We then apply the Gram-Schmidt process to find an orthonormal basis.

$$\vec{w}_1 = \frac{1}{\|\vec{v}_1\|}\vec{v}_1 = \frac{1}{3}\begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \\ 2 \\ 0 \end{bmatrix}$$

$$\vec{w}_2 = \frac{1}{\|\vec{v}_2 - (\vec{w}_1 \cdot \vec{v}_2)\vec{w}_1\|}(\vec{v}_2 - (\vec{w}_1 \cdot \vec{v}_2)\vec{w}_1) = \frac{1}{3}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \\ -2 \\ 1 \end{bmatrix}$$

Note that $\vec{w}_1 \cdot \vec{w}_2 = 0$, that $\|\vec{w}_1\| = \|\vec{w}_2\| = 1$, and that $A\vec{w}_1 = A\vec{w}_2 = 0$, as desired.

(b) Let

$$B = \begin{bmatrix} 2 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

and

$$C = \begin{bmatrix} 0 & 0 \\ 2 & 0 \\ 0 & 0 \\ -1 & 1 \\ 0 & -1 \\ 2 & -4 \end{bmatrix}.$$

Let V be the linear subspace of all the vectors $\vec{v}$ of $\mathbf{R}^6$ which are at the same time in the kernel of B and perpendicular to the image of C. Find an orthonormal basis of V. [HINT: How is V related to $\ker A$?]

If $\vec{v}$ is perpendicular to the image of C then it is in the kernel of C^T; $(\mathrm{im}A)^\perp = \ker(A^T)$. The vectors that are both in the kernel of B and in the kernel of C^T are simply the kernel of A, for which one orthonormal basis is

$$\vec{w}_1 = \frac{1}{3}\begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \\ 2 \\ 0 \end{bmatrix} \quad \vec{w}_2 = \frac{1}{3}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \\ -2 \\ 1 \end{bmatrix}.$$

(4) In this question we will fit a function of the form

$$f(t) = x_1 + x_2 \sin t + x_3 \cos t$$

to the data points $(t, f(t)) = (0,0)$, $(\pi/2, 2)$, $(\pi, 0)$ and $(3\pi/2, -1)$ using least squares.

(a) Find a matrix A and a vector $\vec{b}$ such that the optimal x_1, x_2, x_3 are a least squares approximate solution to $A\vec{x} = \vec{b}$, where

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Substituting in the four values of t and $f(t)$, we get the following set of equations:

$$\begin{aligned} x_1 + x_2 \cdot 0 + x_3 &= 0 \\ x_1 + x_2 + x_3 \cdot 0 &= 2 \\ x_1 + x_2 \cdot 0 - x_3 &= 0 \\ x_1 - x_2 + x_3 \cdot 0 &= -1 \end{aligned}$$

In matrix form, this gives

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \\ -1 \end{bmatrix}.$$

(b) Find a least squares approximate solution to $A\vec{x} = \vec{b}$.

The normal equation for the system is

$$A^T A\vec{x} = A^T\vec{b}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 0 \\ -1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$$

The least squares approximate solution is then

$$\vec{x}^* = \begin{bmatrix} 1/4 \\ 3/2 \\ 0 \end{bmatrix}.$$

(c) Write down the function $f(t)$ of the above form which, according to the least squares method, best fits the given data points.

Since the vector $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ corresponds to the function $x_1 + x_2 \sin t + x_3 \cos t$, wesee that the function of this form that best fits the data points is

$$f^*(t) = \frac{1}{4} + \frac{3}{2} \sin t.$$

(5) (a) Find the area of the triangle in $\mathbf{R}^4$ with vertices

$$\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

Subtracting the third vertex from all three vertex points translates this triangle so that one vertex is at the origin and the other two now have coordinates $\vec{v}_1 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} -1 \\ -2 \\ -1 \\ 1 \end{bmatrix}$. We can now use the fact that the 2-volume of the parallelogram defined by these two vectors is $\sqrt{\det(A^T A)}$ (Fact 6.3.7), where A is the matrix whose columns are $\vec{v}_1$ and $\vec{v}_2$. The area of the triangle defined by these two vectors is then half of this. So we have

$$\begin{aligned}
\text{area} &= \frac{1}{2}\sqrt{\det\left(\begin{bmatrix} 0 & -1 & 0 & 1 \\ -1 & -2 & -1 & 1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ -1 & -2 \\ 0 & -1 \\ 1 & 1 \end{bmatrix}\right)} \\
&= \frac{1}{2}\sqrt{\det\begin{bmatrix} 2 & 3 \\ 3 & 7 \end{bmatrix}} \\
&= \frac{\sqrt{5}}{2}
\end{aligned}$$

(b) Find

$$\det\begin{bmatrix} 1 & b & b & b & b \\ 1 & 1 & b & b & b \\ 1 & 1 & 1 & b & b \\ 1 & 1 & 1 & 1 & b \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Since the determinant is invariant under elementary row operations, we reduce this matrix by subtracting the second row from the first row, the third row from the second, and so on, to obtain

$$\begin{aligned}\det\begin{bmatrix} 1 & b & b & b & b \\ 1 & 1 & b & b & b \\ 1 & 1 & 1 & b & b \\ 1 & 1 & 1 & 1 & b \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} &= \det\begin{bmatrix} 0 & b-1 & 0 & 0 & 0 \\ 1 & 1 & b & b & b \\ 1 & 1 & 1 & b & b \\ 1 & 1 & 1 & 1 & b \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \\ &= \det\begin{bmatrix} 0 & b-1 & 0 & 0 & 0 \\ 0 & 0 & b-1 & 0 & 0 \\ 1 & 1 & 1 & b & b \\ 1 & 1 & 1 & 1 & b \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \\ &\vdots \\ &= \det\begin{bmatrix} 0 & b-1 & 0 & 0 & 0 \\ 0 & 0 & b-1 & 0 & 0 \\ 0 & 0 & 0 & b-1 & 0 \\ 0 & 0 & 0 & 0 & b-1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}\end{aligned}$$

The determinant of this matrix we can read off using Laplace expansion (expansion by minors) or by patterns, and we get

$$\det\begin{bmatrix} 1 & b & b & b & b \\ 1 & 1 & b & b & b \\ 1 & 1 & 1 & b & b \\ 1 & 1 & 1 & 1 & b \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} = \det\begin{bmatrix} 0 & b-1 & 0 & 0 & 0 \\ 0 & 0 & b-1 & 0 & 0 \\ 0 & 0 & 0 & b-1 & 0 \\ 0 & 0 & 0 & 0 & b-1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} = (b-1)^4.$$

(6) Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

(a) Find a diagonal matrix D and an invertible matrix S such that $A = SDS^{-1}$.

The eigenvalues of A are

$$f(\lambda) = \det(A - \lambda I_3) = \det \begin{bmatrix} 1-\lambda & 1 & 2 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & -\lambda \end{bmatrix}$$

$$\Rightarrow f(\lambda) = (1-\lambda)(\lambda^2 - \lambda - 2) = 0$$

$$\Rightarrow \lambda_{1,2,3} = \{1, -1, 2\}$$

Note that we have three distinct eigenvalues, so there is definitely an eigenbasis for this matrix. The corresponding eigenvectors are

$$\lambda = 1 : \ker(A - I_3) = \text{span} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$\lambda = -1 : \ker(A + I_3) = \text{span} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\lambda = 2 : \ker(A - 2I_3) = \text{span} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

The matrix A is diagonal in the basis of eigenvectors; letting S be the change of basis matrix from the eigenbasis to the standard basis (i.e, the matrix whose columns are the eigenvectors of A), we find that $A = SDS^{-1}$, where D is the matrix with the eigenvalues of A alongthe diagonal and zeros elsewhere:

$$S = \begin{bmatrix} 1 & 1 & 2 \\ -2 & 0 & 0 \\ 1 & -1 & 1 \end{bmatrix}, \ D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Order matters here: the order of the columns of S and the order of their corresponding eigenvalues in D must be the same.

(b) Find all matrices B such that $AB = BA$. You may assume that a matrix E satisfies $DE = ED$ if and only if E is diagonal.
[You may leave your answer as a product of explicit matrices and inverses of explicit matrices.]

First of all we note that all matrices A and B must be square and have the same dimension; otherwise either AB or BA will not make sense. Using the fact that a matrix E satisfies $DE = ED$ (i.e. E and D *commute*) if and only if E is diagonal, and that $A = SDS^{-1}$ from the previous part, we see that

$$\begin{aligned} AB = BA &\Leftrightarrow SDS^{-1}B = BSDS^{-1} \\ &\Leftrightarrow DS^{-1}BS = S^{-1}BSD \\ &\Leftrightarrow S^{-1}BS \text{ is diagonal} \end{aligned}$$

So in general, if A is an $n \times n$ and $D = S^{-1}AS$ is diagonal, then the matrices that commute with A are simply the $n \times n$ matrices B such that $B = SB'S^{-1}$ for some $n \times n$ diagonal matrix B'.