

Name: \_\_\_\_\_

**Math 21b Midterm II—Thursday, April 10, 2003**

*Please circle your section:*

Thomas Judson	Thomas Judson	Kalle Karu
Eduardo Saverin (CA)	Rene Shen (CA)	Mark Bandstra (CA)
Jakob Topp (CA)	Albert Wang (CA)	MWF 12-1
MWF 10–11	MWF 11–12	

Ken Chung	Spiro Karigiannis
Nathan Lange (CA)	Jeff Berton (CA)
TuTh 10–11:30	TuTh 11:30–1

Problem Number	Possible Points	Score
1	12	
2	16	
3	16	
4	16	
5	8	
6	16	
7	16	
Total	100	

**Directions—Please Read Carefully!** You have two hours to take this midterm. Pace yourself by keeping track of how many problems you have left to go and how much time remains. you do not have to answer the problems in any particular order, so move to another problem if you find you are stuck or spending too much time on a single problem. To receive full credit on a problem, you will need to justify your answers carefully—unsubstantiated answers will receive little or no credit (except) if the directions for that question specifically say no justification is necessary, such as in the True/False section). Please be sure to write neatly—illegible answers will receive little or no credit. If more space is needed, use the back of the previous page to continue your work. Be sure to make a note of this on the problem page so that the grader knows where to find your answers. You are allowed one 4 by 6 inch file card of notes to use on the examination. No calculators or other aids are allowed. **Good Luck on the Exam!**

1. (12 points) True or False. No justification is necessary, simply circle **T** or **F** for each statement.

**T**   **F**   (a)   If  $A$  is an  $n \times n$  invertible matrix, then  $\det(A^T A) > 0$ .

**T**   **F**   (b)   The subset

$$W = \{f \in C^\infty(\mathbb{R}) : f''(x) + f(x) = x^2\}$$

is a subspace of the linear space of all infinitely differentiable functions  $C^\infty(\mathbb{R})$ .

**T**   **F**   (c)   If  $\mathbf{v} \in \mathbb{R}^n$  and  $W$  is a subspace of  $\mathbb{R}^n$ , then

$$\|\mathbf{v}\|^2 = \|\text{proj}_W \mathbf{v}\|^2 + \|\mathbf{v} - \text{proj}_W \mathbf{v}\|^2.$$

**T**   **F**   (d)   If  $\det(A) \neq \det(B)$ , then two  $n \times n$  matrices  $A$  and  $B$  cannot be similar.

**T**   **F**   (e)   The vectors  $x + 1$ ,  $x - 1$ , and  $x^2 - 1$  are linearly dependent in  $P_2$ , the polynomials of degree less than or equal to 2.

**T**   **F**   (f)   The determinant of

$$A = \begin{bmatrix} 1 & 1000 & 2 & 3 & 4 \\ 5 & 6 & 1000 & 7 & 8 \\ 1000 & 9 & 8 & 7 & 6 \\ 5 & 4 & 3 & 2 & 1000 \\ 1 & 2 & 3 & 1000 & 4 \end{bmatrix}$$

is positive.

2. Let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

(a) (8 points) Find an orthonormal basis for the image of  $A$ .

(b) (8 points) Find the  $QR$  factorization of  $A$ .

3. (a) (8 points) Find the linear function  $y = c_0 + c_1x$  that best fits the following data using least squares.

$x$	-6	-2	1	7
$y$	-1	2	1	6

- (b) (8 points) Let  $A$  be an  $m \times n$  matrix such that  $\text{rank}(A) = n$  and  $\mathbf{b} \in \mathbb{R}^m$ . If  $A = QR$  is the  $QR$  factorization of  $A$ , show that the unique least squares solution of  $A\mathbf{x} = \mathbf{b}$  is given by

$$\hat{\mathbf{x}} = R^{-1}Q^T\mathbf{b}.$$

4. Let  $\mathbf{u}_1 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ ,  $\mathbf{u}_2 = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$  be a basis for  $\mathbb{R}^2$ .

(a) (8 points) If  $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ , find  $c_1$  and  $c_2$  such that  $\mathbf{x} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2$ .

(b) (8 points) If  $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ ,  $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$  is another basis for  $\mathbb{R}^2$ , find  $k_1$  and  $k_2$  such that  $-2\mathbf{u}_1 + 3\mathbf{u}_2 = k_1\mathbf{v}_1 + k_2\mathbf{v}_2$ .

5. (8 points) Use row and column operations to find the determinant of the matrix

$$A = \begin{bmatrix} a_1 + 1 & a_2 & a_3 & \dots & a_n \\ a_1 & a_2 + 1 & a_3 & \dots & a_n \\ a_1 & a_2 & a_3 + 1 & \dots & a_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & a_3 & \dots & a_n + 1 \end{bmatrix}$$

where  $a_1, \dots, a_n$  are real numbers.

6. (a) (8 points) Let  $A$  be an  $n \times n$  skew-symmetric matrix. That is,  $A^T = -A$ . If  $n$  is odd, show that  $A$  cannot be invertible. [*Hint*: Show  $\det A = 0$ .]

- (b) (8 points) Let  $A$  be an  $n \times n$  skew-symmetric matrix and  $\mathbf{x}$  be an  $n \times 1$  column vector. Show that  $\mathbf{x}^T A \mathbf{x} = 0$  for all  $\mathbf{x} \in \mathbb{R}^n$ .

7. (a) (8 points) Let  $V$  be the subspace of  $\mathbb{R}^4$  spanned by

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 3 \\ -2 \end{bmatrix}.$$

Find a basis for  $V^\perp$ .

- (b) (8 points) Let  $A$  be an  $m \times n$  matrix. Show that  $\ker(A^T A) = \ker(A)$ . [*Hint:* First show that if  $\mathbf{x} \in \ker(A^T A)$ , then  $A\mathbf{x}$  is in both  $\text{image}(A)$  and  $\ker(A^T)$ .]