A bit of review for the second Math 21b Midterm on Monday, April 15, 2002

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Dot product - If $v = (v_1, v_2, \ldots v_n)$ and $w = (w_1, w_2, \ldots, w_n)$, then their dot product, $v \cdot w = v_1w_1 + v_2w_2 + \ldots + v_nw_n$.

Orthogonality - two vectors are called orthogonal (=perpendicular) if their dot product is zero.

Length - the length of a vector v is $||v|| = \sqrt{v \cdot v}$. A unit vector has length 1.

Triangle inequality - $||v + w|| \leq ||v|| + ||w||$.

Orthonormal vectors - a set of unit vectors that are all perpendicular: $v_i \cdot v_j = 1$ if $i = j$ and $v_i \cdot v_j = 0$ if $i \neq j$. n orthonormal vectors in $\mathbf{R}^n$ are linearly independent and form a basis.

Orthonormal basis - a set of orthogonal unit vectors $v_1, \ldots, v_n$ that form a basis for the space. For any x in $\mathbf{R}^n$, there is a unique way to write x as a linear combination of $v_1, \ldots, v_n$. It is: $x = (v_1 \cdot x)v_1 + \ldots + (v_n \cdot x)v_n$.

Orthogonal complement of a subset V of $\mathbf{R}^n$ - denoted $V^\perp$ is the set of vectors in $\mathbf{R}^n$, that are orthogonal to all vectors in V; it is a subspace if V is a subspace: $V^\perp = \{x \in \mathbf{R}^n \mid v \cdot x = 0, \text{ for all } v \in V\}$.

Orthogonal projection onto a subspace V with orthonormal basis $v_1, \ldots, v_m$ - for all x in $\mathbf{R}^n$, there exists a unique w in V such that $(w - x)$ is in $V^\perp$. The vector w is the orthogonal projection of x onto V; the formula for this linear transformation is $\text{proj}_V(x) = (v_1 \cdot x)v_1 + \ldots + (v_m \cdot x)v_m$.

Cauchy Schwartz inequality - $|x \cdot y| \leq ||x||||y||$.

Angle - the angle, α, between vectors x and y is equal to $\alpha = \arccos(x \cdot y)/(||x|| \cdot ||y||)$.

Gram-Schmidt Process - a way of making a set of linearly independent vectors into an orthonormal set of vectors without changing the actual space they span; the steps in the process can be found on page 197 of the text and basically amount to the repetition of two steps for each vector:

1. Make the vector perpendicular to all the other vectors already looked at.
2. Make it a unit vector

QR Factorization - $A = QR$ is obtained in the Gram-Schmidt Process. the exact content of each matrix can be found on page 198 of the text.

Transpose - the transpose of a matrix reflects the elements across the main diagonal; in other words, elements a_{ij} and a_{ji} are interchanged; notice that we can now write $v \cdot w$ as $v^T w$. The transpose has the following properties:

1.$(AB)^T = B^T A^T$ (note that the order switches!)
2.$(A^T)^{-1} = (A^{-1})^T$.
3.$(A^T)^T = A$.

Symmetric - a matrix A is called symmetric if $A^T = A$.

Skew-symmetric - a matrix A is called skew-symmetric if $A^T = -A$.

Orthogonal Transformation - a transformation T from $\mathbf{R}^n$ to $\mathbf{R}^n$ that preserves length and angles. T is orthogonal if and only $T(e_1), \ldots T(e_n)$ are an orthonormal basis of R^n. This is equivalent to saying that A is an orthogonal matrix if and only if its columns are an orthonormal basis; products and inverses of orthogonal matrices are orthogonal; the following statements about an $n \times n$ matrix A are equivalent. Either they are all true or they are all false):

1. A is an orthogonal matrix
2. $A(x)$ preserves the length: $||A(x)|| = ||x||$.
3. The columns of A form an orthonormal basis for R^n
4. $A^T A = I_n$.

5. $A^{-1} = A^T$

Orthogonal projection onto V - if V is a subspace of $\mathbf{R}^n$ with orthonormal basis $v_1, \ldots, v_m$ and A is the matrix whose columns are $v_1, \ldots, v_m$, then the matrix representing the orthogonal projection onto V is AA^T (note the order).

Determinant - the determinant is a number assigned to a square matrix.

Determinant of a 1×1 matrix - $\det[a] = a$.
Determinant of a 2×2 matrix - $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$.
Determinant of a 3x3 matrix - $\det \begin{bmatrix} u & v & w \end{bmatrix} = u \cdot (v \times w)$.

In general, $\det(A) = \sum_{\text{even patterns}\pi} a_{1\pi(1)} \cdots a_{n\pi(n)} - \sum_{\text{odd patterns}\pi} a_{1\pi(1)} \cdots a_{n\pi(n)}$. A pattern is even if an even number of inversions occur, otherwise it is odd.

Inversion - an inversion occurs when one element of a pattern is below and to the left of another element of a pattern.

Effects of row operations on the determinant: A is an $n \times n$ matrix and B is the matrix obtained by performing some operation on A. If the operation is swapping two rows, $\det(B) = -\det(A)$ if the operation is dividing a row of A by k, then $\det(B) = (1/k)\det(A)$; if the operation is adding a scalar multiple of one row to another, then $\det(B) = \det(A)$ this can be summarized by saying that in the process of row reducing A to the identity matrix, if s is number of swaps (of rows) made and $k_1, \ldots k_r$ are the scalars that individual rows were divided by, then $\det(A) = (-1)^s k_1 k_2 \cdots k_r$.

An $n \times n$ matrix A is invertible if and only if $\det(A) \neq 0$. $\det(AB) = \det(A)\det(B)$; note, that the determinant of the sum of two matrices is not necessarily equivalent to the sum of the determinants of two matrices $\det(A^{-1}) = (\det(A))^{-1} = 1/(\det(A))$.

Minor - if A is an $n \times n$ matrix, then A_{ij} is defined as the $(n-1) \times (n-1)$ matrix formed by eliminating the i'th row of A and the j'th column of A.

Lagrange expansion - The determinant of an $n \times n$ matrix can be computed by the formula $\det(A) = a_{11}\det(A_{11}) - a_{12}\det(A_{12}) + \cdots + (-1)^{1+n} a_{1n}\det(A_{1n})$. There are similar expansion for other columns or rows.

Determinant of an orthogonal matrix is ± 1, if it is 1 then the matrix is called a **rotation matrix**.

Volume of a Parallelepiped - if $v_1, \cdots, v_n$ are vectors in $\mathbf{R}^n$, then the volume of the parallelepiped spanned by them is the determinant of the matrix whose column vectors are $v_1, \ldots, v_n$; if you only have vectors $v_1, , v_k$ in $\mathbf{R}^n$, where $k \leq n$, the volume of the k-parallelepiped is $(det(A^T A)$ where A is the matrix whose columns are $v_1, \ldots, v_k$; If $k = n$, this is exactly the same as the first definition.

The determinant from the QR decomposition - if $v_1, \ldots, v_n$ are the columns of matrix A, then $|\det(A)| = ||v_1|| ||v_2 - \text{proj}_{V_1} v_2|| \cdots ||v_n - \text{proj}_{V_n} - v_n||$, where V_i is the space spanned by vectors $v_1, \ldots, v_i$; note that this says $|\det(A)| = |\det(R)|$, where R is the matrix from the QR factorization of $A = QR$.

Expansion factor - if Ω is some region and T is a linear transformation represented by the matrix A, then
Volume of $T(\Omega))/(\text{Volume of } \Omega) = |\det(A)|$.

Cramer's Rule: to solve the system $Ax = b$, the i'th coordinate of the vector x, $x_i = \det(A_i)/\det(A)$, where A_i is the matrix formed by replacing the ith column of A by the vector b.

State vector - a vector x that contains all the information of what is happening in a dynamical system at a time t.

Phase portrait - a picture of what happens at present, past, and future times of a linear dynamical system $x \mapsto Ax$; it plots the state vectors $x(t) = A(x(t-1))$ for all integers t.

Eigenvector - an eigenvector of the matrix A is a nonzero vector v such that $Av = \lambda v$ for some value λ.

Eigenvalue - an eigenvalue of A is a real or complex number λ for which $A(v) = \lambda v$ for a nonzero vector v.

Eigenvalues of an orthogonal matrix - the eigenvalues of an orthogonal matrix satisfy $|\lambda| = 1$.

Very important to study - go through Summary 7.1.4 on page 299; memorize it if necessary, but make sure you understand roughly why each of the statements are equivalent.

Characteristic polynomial - $f_A(\lambda) = det(\lambda I_n - A)$. The eigenvalues of an $n \times n$ matrix A are the zeroes of this function; λ is an eigenvalue of A if and only if $\det(\lambda I_n - A) = 0$.

Eigenvalues of a triangular matrix A - are the diagonal entries or the matrix. The determimant is the product of the diagonal entries.

Trace - the sum of the diagonal elements of an $n \times n$ matrix A is called the trace of A, written $\mathrm{tr}(A)$.

Eigenvalues of a 2x2 matrix - The characteristic polynomial is $f_A(\lambda) = \lambda^2 - \mathrm{tr}(A) + \det(A)$.

Algebraic multiplicity - eigenvalue λ_0 has algebraic multiplicity k $iff_A(\lambda) = (\lambda - \lambda_0)g(\lambda)$ for a polynomial $g(\lambda)$ that λ_0 is not a root of; in other words, λ_0 is a root of multiplicity k of $f_A(\lambda)$.

An $n \times n$ matrix has at most n eigenvalues, even if they are counted with their algebraic multiplicities; if n is odd, then an $n \times n$ matrix has at least one eigenvalue.

Eigenspace - the kernel of the matrix $\lambda I_n - A$ is the eigenspace associated with λ and written E_λ; this space consists of all solutions v to the equation $A(v) = \lambda v$.

Geometric multiplicity - if λ is an eigenvalue of a matrix A, then the dimension of E_λ is called the geometric multiplicity of λ; The geometric multiplicity of $\lambda) \leq$ algebraic multiplicity of λ.

Eigenbasis - if A is an $n \times n$ matrix, a basis of $\mathbf{R}^n$ consisting of eigenvectors of A is called an eigenbasis for A.

If $v_1, v_2, , v_m$ are eigenvectors of an $n \times n$ matrix A with distinct eigenvalues $\lambda_1, \lambda_2, \ldots \lambda_m$, hen the vectors are linearly independent. If A has n distinct eigenvalues then there is an eigenbasis for A found by choosing an eigenvector for each eigenvalue. If the geometric multiplicities of the eigenvalues of A add up to n then there is an eigenbasis for A found by choosing a basis of each eigenspace and combining these vectors.

Complex numbers - $z = a + ib$, where $i = \sqrt{-1}$; $a = \mathrm{Re}(z)$ is called the "real part" of z, and $b = \mathrm{Im}(z)$ is called the "imaginary part" of z.

The length of z, $|z|$, is called the modulus of z and the angle z makes, is called the argument of z; complex numbers operate in the following ways: $(a + ib) + (c + id) = (a + c) + i(b + d)$ $(a + ib)(c + id) = (ac - bd) + i(ad + bc)$.

Polar form - the polar form of the complex number z is $z = r(\cos(\phi) + isin(\phi)$.

$|zw| = |z||w|$. $\arg(zw) = \arg(z) + \arg(w)(\mathrm{modulo} 2\pi)$.

DeMoivre's formula - $(cos(\phi) + i\sin(\phi))^n = \cos(n\phi) + i\sin(n\phi)$.

Fundamental theorem of algebra - any polynomial $p(\lambda)$ with complex coefficients can be written as a product of linear factors: $p(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n)$ for some complex numbers (not necessarily distinct) $\lambda_1, \ldots, \lambda_n$, so a polynomial of degree n has exactly n roots when counted with multiplicity.

Complex eigenvalues and eigenvectors - a complex $n \times n$ matrix has n complex eigenvalues if eigenvalues are counted with their algebraic multiplicities.

If the eigenvalues are $\lambda_1, \ldots, \lambda_n$, listed with their algebraic multiplicities, then $\mathrm{tr}(A) = \lambda_1 + \ldots + \lambda_n$ and $\det(A) = \lambda_1 \cdots \lambda_n$.

Stable equilibrium - In a dynamical system $x(t+1) = Ax(t)$, 0 is an asymptotically stable equilibrium $x(t) \to 0$ as $t \to \infty$. This is true if and only if the modulus of every complex eigenvalue of A is less than 1. If $|\lambda| = 1$, then the trajectories are ellipses or lines. If $|\lambda| > 1$ then the trajectory in general spirals outward. See review Fact 7.6.2 on page 356.

Practice Problems. You certainly do not have to do all of these in order to be prepared for the midterm, but it would not be a bad idea to look through them. Also, all the problems are odd-numbered problems, so their answers should be in the back of the book)

Section 5.1: 5, 13, 17, 19, 27, 31
Section 5.2: 7, 21, 33, 35, 41
Section 5.3: 7, 13, 17, 25, 31
Section 5.4: 5, 7, 23, 25, 33
Section 6.1: 7, 19, 31, 39
Section 6.2: 3, 9, 17, 19, 25, 31, 39
Section 6.3: 7, 13, 21, 23, 27, 31
Section 7.1: 5, 17, 23, 27, 35, 39, 41
Section 7.2: 11, 15, 19, 27, 29, 37
Section 7.3: 5, 9, 19, 25, 35, 37, 41, 45
Section 7.4: 5, 11, 19, 23, 35, 37
Section 7.5: 3, 9, 15, 21, 25, 29, 35, 41
Section 7.6: 7, 15, 21, 29, 41

More Questions/Problems

1. Consider the matrix $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$.

a) What are the eigenvalues and eigenvectors of A?
b) $x = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$ as a linear combination of the eigenvectors of A.
c) Without technology, and without multiplying A seven times by itself, what is $A^7(x)$.

2. Consider the matrix $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 5 & 4 \\ -1 & -2 & 0 \end{bmatrix}$.

solve the equation $A(x) = b$ for the following choices of b using Cramer's Rule

a) $b = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}$

b) $b = \begin{bmatrix} -4 \\ -2 \\ 5 \end{bmatrix}$.

3. Find all real values of a so that the vectors $v_1 = \begin{bmatrix} a \\ 1 \\ -1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 3a \\ -4 \end{bmatrix}$, and $v_3 = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}$ form a linearly independent set.

4. Given an $n \times n$ matrix A and a nonzero vector x in R^n, complete the following:

a) The nonzero vector x is called an eigenvector of A if ...?
b) The scalar discussed in (a) is called ...?
c) The eigenspace corresponding to a given scalar mentioned in b) is ...?

5. Given the matrix $A = \begin{bmatrix} 3 & 2 & 2 \\ 1 & 4 & 1 \\ -2 & -4 & -1 \end{bmatrix}$. Write the characteristic polynomial of A and find the eigenvalues. Find eigenvectors corresponding to each eigenvalue.