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**Dot product** - If  $v = (v_1, v_2, \dots, v_n)$  and  $w = (w_1, w_2, \dots, w_n)$ , then their dot product,  $v \cdot w = v_1 w_1 + v_2 w_2 + \dots + v_n w_n$ .

**Orthogonality** - two vectors are called orthogonal (=perpendicular) if their dot product is zero.

**Length** - the length of a vector  $v$  is  $\|v\| = \sqrt{v \cdot v}$ . A unit vector has length 1.

**Triangle inequality** -  $\|v + w\| \leq \|v\| + \|w\|$ .

**Orthonormal vectors** - a set of unit vectors that are all perpendicular:  $v_i \cdot v_j = 1$  if  $i = j$  and  $v_i \cdot v_j = 0$  if  $i \neq j$ .  $n$  orthonormal vectors in  $\mathbf{R}^n$  are linearly independent and form a basis.

**Orthonormal basis** - a set of orthogonal unit vectors  $v_1, \dots, v_n$  that form a basis for the space. For any  $x$  in  $\mathbf{R}^n$ , there is a unique way to write  $x$  as a linear combination of  $v_1, \dots, v_n$ . It is:  $x = (v_1 \cdot x)v_1 + \dots + (v_n \cdot x)v_n$ .

**Orthogonal complement of a subset  $V$  of  $\mathbf{R}^n$**  - denoted  $V^\perp$  is the set of vectors in  $\mathbf{R}^n$ , that are orthogonal to all vectors in  $V$ ; it is a subspace if  $V$  is a subspace:  $V^\perp = \{x \in \mathbf{R}^n \mid v \cdot x = 0, \text{ for all } v \in V\}$ .

**Orthogonal projection onto a subspace  $V$  with orthonormal basis  $v_1, \dots, v_m$**  - for all  $x$  in  $\mathbf{R}^n$ , there exists a unique  $w$  in  $V$  such that  $(w - x)$  is in  $V^\perp$ . The vector  $w$  is the orthogonal projection of  $x$  onto  $V$ ; the formula for this linear transformation is  $\text{proj}_V(x) = (v_1 \cdot x)v_1 + \dots + (v_m \cdot x)v_m$ .

**Cauchy Schwartz inequality** -  $|x \cdot y| \leq \|x\| \|y\|$ .

**Angle** - the angle,  $\alpha$ , between vectors  $x$  and  $y$  is equal to  $\alpha = \arccos(x \cdot y) / (\|x\| \cdot \|y\|)$ .

**Gramm-Schmitt Process** - a way of making a set of linearly independent vectors into an orthonormal set of vectors without changing the actual space they span; the steps in the process can be found on page 201 of the text and basically amount to the repetition of two steps for each vector:

1. Make the vector perpendicular to all the other vectors already looked at.
2. Make it a unit vector

**QR Factorization** -  $A = QR$  is obtained in the Gramm-Schmitt Process. the exact content of each matrix can be found on page 202 of the text.

**Transpose** - the transpose of a matrix reflects the elements across the main diagonal; in other words, elements  $a_{ij}$  and  $a_{ji}$  are interchanged; notice that we can now write  $v \cdot w$  as  $v^T w$ . The transpose has the following properties:

1.  $(AB)^T = B^T A^T$  (note that the order switches!)
2.  $(A^T)^{-1} = (A^{-1})^T$ .
3.  $(A^T)^T = A$ .

**Symmetric** - a matrix  $A$  is called symmetric if  $A^T = A$ .

**Skew-symmetric** - a matrix  $A$  is called skew-symmetric if  $A^T = -A$ .

**Orthogonal Transformation** - a transformation  $T$  from  $\mathbf{R}^n$  to  $\mathbf{R}^n$  that preserves length and angles.  $T$  is orthogonal if and only if  $T(e_1), \dots, T(e_n)$  are an orthonormal basis of  $\mathbf{R}^n$ . This is equivalent to saying that  $A$  is an orthogonal matrix if and only if its columns are an orthonormal basis; products and inverses of orthogonal matrices are orthogonal; the following statements about an  $n \times n$  matrix  $A$  are equivalent. Either they are all true or they are all false):

1.  $A$  is an orthogonal matrix
2.  $A(x)$  preserves the length:  $\|A(x)\| = \|x\|$ .
3. The columns of  $A$  form an orthonormal basis for  $\mathbf{R}^n$
4.  $A^T A = I_n$ .

5.  $A^{-1} = A^T$

**Orthogonal projection onto  $V$**  - if  $V$  is a subspace of  $\mathbf{R}^n$  with orthonormal basis  $v_1, \dots, v_m$  and  $A$  is the matrix whose columns are  $v_1, \dots, v_m$ , then the matrix representing the orthogonal projection onto  $V$  is  $AA^T$  (note the order).

**Determinant** - the determinant is a number assigned to a square matrix.

Determinant of a  $1 \times 1$  matrix -  $\det[a] = a$ .

Determinant of a  $2 \times 2$  matrix -  $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$ .

Determinant of a  $3 \times 3$  matrix -  $\det \begin{bmatrix} u & v & w \end{bmatrix} = u \cdot (v \times w)$ .

In general,  $\det(A) = \sum_{\text{even patterns } \pi} a_{1\pi(1)} \cdots a_{n\pi(n)} - \sum_{\text{odd patterns } \pi} a_{1\pi(1)} \cdots a_{n\pi(n)}$ . A pattern is even if an even number of inversions occur, otherwise it is odd.

**Inversion** - an inversion occurs when one element of a pattern is below and to the left of another element of a pattern.

**Effects of row operations on the determinant:**  $A$  is an  $n \times n$  matrix and  $B$  is the matrix obtained by performing some operation on  $A$ . If the operation is swapping two rows,  $\det(B) = -\det(A)$  if the operation is dividing a row of  $A$  by  $k$ , then  $\det(B) = (1/k)\det(A)$ ; if the operation is adding a scalar multiple of one row to another, then  $\det(B) = \det(A)$  this can be summarized by saying that in the process of row reducing  $A$  to the identity matrix, if  $s$  is number of swaps (of rows) made and  $k_1, \dots, k_r$  are the scalars that individual rows were divided by, then  $\det(A) = (-1)^s k_1 k_2 \cdots k_r$ .

An  $n \times n$  matrix  $A$  is invertible if and only if  $\det(A) \neq 0$ .  $\det(AB) = \det(A)\det(B)$ ; note, that the determinant of the sum of two matrices is not necessarily equivalent to the sum of the determinants of two matrices  $\det(A^{-1}) = (\det(A))^{-1} = 1/(\det(A))$ .

**Minor** - if  $A$  is an  $n \times n$  matrix, then  $A_{ij}$  is defined as the  $(n-1) \times (n-1)$  matrix formed by eliminating the  $i$ 'th row of  $A$  and the  $j$ 'th column of  $A$ .

**Lagrange expansion** - The determinant of an  $n \times n$  matrix can be computed by the formula  $\det(A) = a_{11}\det(A_{11}) - a_{12}\det(A_{12}) + \cdots + (-1)^{1+n}a_{1n}\det(A_{1n})$ . There are similar expansion for other columns or rows.

**Determinant of an orthogonal matrix** is  $\pm 1$ , if it is 1 then the matrix is called a **rotation matrix**.

**Volume of a Parallelepiped** - if  $v_1, \dots, v_n$  are vectors in  $\mathbf{R}^n$ , then the volume of the parallelepiped spanned by them is the determinant of the matrix whose column vectors are  $v_1, \dots, v_n$ ; if you only have vectors  $v_1, \dots, v_k$  in  $\mathbf{R}^n$ , where  $k \leq n$ , the volume of the  $k$ -parallelepiped is  $(\det(A^T A))^{1/2}$  where  $A$  is the matrix whose columns are  $v_1, \dots, v_k$ ; If  $k = n$ , this is exactly the same as the first definition.

**The determinant from the QR decomposition** - if  $v_1, \dots, v_n$  are the columns of matrix  $A$ , then  $|\det(A)| = \|v_1\| \|v_2 - \text{proj}_{V_1} v_2\| \cdots \|v_n - \text{proj}_{V_{n-1}} v_n\|$ , where  $V_i$  is the space spanned by vectors  $v_1, \dots, v_i$ ; note that this says  $|\det(A)| = |\det(R)|$ , where  $R$  is the matrix from the QR factorization of  $A = QR$ .

**Expansion factor** - if  $\Omega$  is some region and  $T$  is a linear transformation represented by the matrix  $A$ , then

Volume of  $T(\Omega)$  / (Volume of  $\Omega$ ) =  $|\det(A)|$ .

**Cramer's Rule:** to solve the system  $Ax = b$ , the  $i$ 'th coordinate of the vector  $x$ ,  $x_i = \det(A_i)/\det(A)$ , where  $A_i$  is the matrix formed by replacing the  $i$ th column of  $A$  by the vector  $b$ .

**State vector** - a vector  $x$  that contains all the information of what is happening in a dynamical system at a time  $t$ .

**Phase portrait** - a picture of what happens at present, past, and future times of a linear dynamical system  $x \mapsto Ax$ ; it plots the state vectors  $x(t) = A(x(t-1))$  for all integers  $t$ .

**Eigenvector** - an eigenvector of the matrix  $A$  is a nonzero vector  $v$  such that  $Av = \lambda v$  for some value  $\lambda$ .

**Eigenvalue** - an eigenvalue of  $A$  is a real or complex number  $\lambda$  for which  $A(v) = \lambda v$  for a nonzero vector  $v$ .

**Eigenvalues of an orthogonal matrix** - the eigenvalues of an orthogonal matrix satisfy  $|\lambda| = 1$ .

Very important to study - go through Summary 6.1.4 on page 302; memorize it if necessary, but make sure you understand roughly why each of the statements are equivalent.

**Characteristic polynomial** -  $f_A(\lambda) = \det(\lambda I_n - A)$ . The eigenvalues of an  $n \times n$  matrix  $A$  are the zeroes of this function;  $\lambda$  is an eigenvalue of  $A$  if and only if  $\det(\lambda I_n - A) = 0$ .

**Eigenvalues of a triangular matrix**  $A$  - are the diagonal entries of the matrix. The determinant is the product of the diagonal entries.

**Trace** - the sum of the diagonal elements of an  $n \times n$  matrix  $A$  is called the trace of  $A$ , written  $\text{tr}(A)$ .

**Eigenvalues of a 2x2 matrix** - The characteristic polynomial is  $f_A(\lambda) = \lambda^2 - \text{tr}(A)\lambda + \det(A)$ .

**Algebraic multiplicity** - eigenvalue  $\lambda_0$  has algebraic multiplicity  $k$  if  $f_A(\lambda) = (\lambda - \lambda_0)^k g(\lambda)$  for a polynomial  $g(\lambda)$  that  $\lambda_0$  is not a root of; in other words,  $\lambda_0$  is a root of multiplicity  $k$  of  $f_A(\lambda)$ .

An  $n \times n$  matrix has at most  $n$  eigenvalues, even if they are counted with their algebraic multiplicities; if  $n$  is odd, then an  $n \times n$  matrix has at least one eigenvalue.

**Eigenspace** - the kernel of the matrix  $\lambda I_n - A$  is the eigenspace associated with  $\lambda$  and written  $E_\lambda$ ; this space consists of all solutions  $v$  to the equation  $A(v) = \lambda v$ .

**Geometric multiplicity** - if  $\lambda$  is an eigenvalue of a matrix  $A$ , then the dimension of  $E_\lambda$  is called the geometric multiplicity of  $\lambda$ ; The geometric multiplicity of  $\lambda \leq$  algebraic multiplicity of  $\lambda$ .

**Eigenbasis** - if  $A$  is an  $n \times n$  matrix, a basis of  $\mathbf{R}^n$  consisting of eigenvectors of  $A$  is called an eigenbasis for  $A$ .

If  $v_1, v_2, \dots, v_m$  are eigenvectors of an  $n \times n$  matrix  $A$  with distinct eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_m$ , then the vectors are linearly independent. If  $A$  has  $n$  distinct eigenvalues then there is an eigenbasis for  $A$  found by choosing an eigenvector for each eigenvalue. If the geometric multiplicities of the eigenvalues of  $A$  add up to  $n$  then there is an eigenbasis for  $A$  found by choosing a basis of each eigenspace and combining these vectors.

**Complex numbers** -  $z = a + ib$ , where  $i = \sqrt{-1}$ ;  $a = \text{Re}(z)$  is called the "real part" of  $z$ , and  $b = \text{Im}(z)$  is called the "imaginary part" of  $z$ .

The length of  $z$ ,  $|z|$ , is called the modulus of  $z$  and the angle  $z$  makes, is called the argument of  $z$ ; complex numbers operate in the following ways:  $(a + ib) + (c + id) = (a + c) + i(b + d)$   $(a + ib)(c + id) = (ac - bd) + i(ad + bc)$ .

**Polar form** - the polar form of the complex number  $z$  is  $z = r(\cos(\phi) + i\sin(\phi))$ .

$|zw| = |z||w|$ .  $\arg(zw) = \arg(z) + \arg(w) \pmod{2\pi}$ .

**DeMoivre's formula** -  $(\cos(\phi) + i\sin(\phi))^n = \cos(n\phi) + i\sin(n\phi)$ .

**Fundamental theorem of algebra** - any polynomial  $p(\lambda)$  with complex coefficients can be written as a product of linear factors:  $p(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n)$  for some complex numbers (not necessarily distinct)  $\lambda_1, \dots, \lambda_n$ , so a polynomial of degree  $n$  has exactly  $n$  roots when counted with multiplicity.

**Complex eigenvalues and eigenvectors** - a complex  $n \times n$  matrix has  $n$  complex eigenvalues if eigenvalues are counted with their algebraic multiplicities.

If the eigenvalues are  $\lambda_1, \dots, \lambda_n$ , listed with their algebraic multiplicities, then  $\text{tr}(A) = \lambda_1 + \dots + \lambda_n$  and  $\det(A) = \lambda_1 \cdots \lambda_n$ .

**Stable equilibrium** - In a dynamical system  $x(t+1) = Ax(t)$ , 0 is an asymptotically stable equilibrium  $x(t) \rightarrow 0$  as  $t \rightarrow \infty$ . This is true if and only if the modulus of every complex eigenvalue of A is less than 1. If  $|\lambda| = 1$ , then the trajectories are ellipses or lines. If  $|\lambda| > 1$  then the trajectory in general spirals outward. See review Fact 6.5.3 on page 361.

**Practice Problems.** You certainly do not have to do all of these in order to be prepared for the midterm, but it would not be a bad idea to look through them. Also, all the problems are odd-numbered problems, so their answers should be in the back of the book)

Section 4.1: 5, 13, 17, 19, 27, 31

Section 4.2: 7, 21, 33, 35, 41

Section 4.3: 7, 13, 17, 25, 31

Section 4.4: 5, 7, 23, 25, 33

Section 5.1: 7, 19, 31, 39

Section 5.2: 3, 9, 17, 19, 25, 31, 39

Section 5.3: 7, 13, 21, 23, 27, 31

Section 6.1: 5, 17, 23, 27, 35, 39, 41

Section 6.2: 11, 15, 19, 27, 29, 37

Section 6.3: 5, 9, 19, 25, 35, 37, 41, 45

Section 6.4: 5, 11, 19, 23, 35, 37

Section 6.5: 3, 9, 15, 21, 25, 29, 35, 41

### More Questions/Problems

1. Consider the matrix  $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$ .

a) What are the eigenvalues and eigenvectors of A?

b)  $x = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$  as a linear combination of the eigenvectors of A.

c) Without technology, and without multiplying A seven times by itself, what is  $A^7(x)$ .

2. Consider the matrix  $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 5 & 4 \\ -1 & -2 & 0 \end{bmatrix}$ .

solve the equation  $A(x) = b$  for the following choices of  $b$  using Cramer's Rule

a)  $b = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}$

b)  $b = \begin{bmatrix} -4 \\ -2 \\ 5 \end{bmatrix}$ .

3. Find all real values of  $a$  so that the vectors  $v_1 = \begin{bmatrix} a \\ 1 \\ -1 \end{bmatrix}$ ,  $v_2 = \begin{bmatrix} 0 \\ 3a \\ -4 \end{bmatrix}$ , and  $v_3 = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}$  form a linearly independent set.

4. Given an  $n \times n$  matrix A and a nonzero vector  $x$  in  $\mathbb{R}^n$ , complete the following:

a) The nonzero vector  $x$  is called an eigenvector of A if ...?

b) The scalar discussed in (a) is called ...?

c) The eigenspace corresponding to a given scalar mentioned in b) is ...?

5. Given the matrix  $A = \begin{bmatrix} 3 & 2 & 2 \\ 1 & 4 & 1 \\ -2 & -4 & -1 \end{bmatrix}$ . Write the characteristic polynomial of A and find the eigenvalues. Find eigenvectors corresponding to each eigenvalue.