Math 21b.
Review for Final Exam

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Spring 2003

General Information

- The exam is on Thursday, May 15 from 2:15 am to 5:15 pm in Jefferson 250. Please check with the registrar if you have a conflict.

- You have 3 hours and there will be approximately 10 questions, including some true/false questions. There will be one or two questions asking for proofs.

- You are permitted on sheet of 8-1/2 by 11 inch notebook paper for notes (both sides).

- No calculators allowed.

- You will be held responsible for one of the following (but not both) on the final exam.

    - Section 9.3 and Sections 10.1–10.3.
    - Markov chains.

Course Wide Review Sessions

- Monday, May 5 at 7:30-9:00 pm in Science Center Auditorium D.

- Friday, May 9 at 3:00-4:30 pm in Science Center Auditorium D.

- Monday, May 12 at 3:00-4:30 pm in Science Center Auditorium D.

We will attempt to videotape all of the review sessions. Videotapes will be on reserve in the Cabot Science Library.

Office Hours and CA Office Hours

You should check with your individual section leader to find out office hours during the reading period. I will hold the following office hours.

- Monday, May 5 at 3–4 PM.

- Tuesday, May 6 at 5–6 PM.

- Wednesday, May 7 at 1–2 PM.

- Thursday, May 8 at 2–3 PM.

- Friday, May 9 at 1–2 PM.

- Monday, May 12 at 1–2 PM.

- Tuesday, May 13 at 3–4 PM.

- Wednesday, May 14 at 3–4 PM.

- Thursday, May 15 at 12–1 PM.

The Course Assistants will hold the following office hours.

- Monday, May 5

- Tuesday, May 6
  4:30–6:30pm, Loker Commons (Eduardo Saverin)
  6:30–7:30pm, Loker Commons (Jeff Berton)
  7:30–8:30pm Loker Commons (Albert Wang)
  9–10pm, Science Center Computer Lab (Rene Shen)

- Wednesday, May 7
  10am–12pm, Loker Commons (Jakub Topp)
  2–3 pm, Loker Commons (Jeff Berton)
  3–4pm, SC 310 (Mark Bandstra)
  8–9pm, Science Center Computer Lab (Rene Shen)

- Thursday, May 8
  1–3pm, Loker Commons (Roger Hong)
  7–8pm, Loker Commons (Albert Wang)

- Sunday, May 11
  7–8pm, SC 411 (Nathan Lange)

- Monday, May 12
  3–4pm, Loker Commons (Mark Bandstra)

- Wednesday, May 14
  7–8pm, the Greenhouse (Nathan Lange)

Chapter 1. Linear Equations

Solving Linear Systems

- You should understand how systems of linear equations can be represented in a matrix.

- You should understand and be able to use elementary row operations to compute the row echelon form and the reduced row echelon form of a matrix.

- You should be able to use Gauss-Jordan elimination to solve linear systems.

- You should be able to interpret solutions geometrically in $\mathbb{R}^2$ and $\mathbb{R}^3$.

Qualitative Facts about Solutions

- You should be able to use the reduced row-echelon form of the augmented matrix to find the number of solutions of a linear system.

- You should be able to apply the definition of the rank of a matrix.

- You should understand that any underdetermined homogeneous system of linear equations must have a nontrivial solution.

Important Fact

Consider a system of m linear equations in n variables. Suppose the $m \times n$ coefficient matrix is A. Then

1. $\mathrm{rank}(A) \leq m$ and $\mathrm{rank}(A) \leq n$.

2. If $\mathrm{rank}(A) = m$, then the system is consistent.

3. If $\mathrm{rank}(A) = n$, then the system has at most one solution.

4. *Undetermined Systems.* if $\mathrm{rank}(A) < n$, then the system has either no solution or an infinite number of solutions.

5. A linear system of n equations in n unknowns has a unique solution if and only if $\mathrm{rank}(A) = n$.

Representing Linear Systems and Their Solutions

- You should be able to compute the product of $A\mathbf{x}$ in terms of the columns or rows of A.

- You should be able represent a linear system in vector or matrix form.

- You should be able to represent all nontrivial solutions of a homogeneous system as a linear combination of vectors. For example, the solution of the system

$$
\begin{aligned}
3x_1 + 2x_2 - x_3 - x_4 &= 0 \\
2x_1 + 3x_2 + 6x_3 + 2x_4 &= 0 \\
x_1 - x_2 - 7x_3 - 3x_4 &= 0
\end{aligned}
$$

is $x_1 = 3s + (7/5)t$, $x_2 = -4s - (8/5)t$, $x_3 = s$, and $x_4 = t$. We can also write the solution as a linear combination of vectors,

$$
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = s \begin{pmatrix} 1 \\ -4 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 7/5 \\ -8/5 \\ 0 \\ 1 \end{pmatrix},
$$

called the *vector form of the general solution.*

Chapter 2. Linear Transformations

Linear Transformations

- You should be able to apply the concept of a linear transformation. A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is a *linear transformation* or *linear map* if there exists and $m \times n$ matrix A such that

$$
T : \mathbf{x} \mapsto A\mathbf{x}
$$

for all $\mathbf{x} \in \mathbb{R}^n$. Equivalently, T satisfies the properties

$$
\begin{aligned}
T(\mathbf{x} + \mathbf{y}) &= T(\mathbf{x}) + T(\mathbf{y}); \\
T(\alpha\mathbf{x}) &= \alpha T(\mathbf{x}).
\end{aligned}
$$

- You should be able to interpret simple linear transformations geometrically. For example, we may ask what happens to the unit square under the following transformations:

$$
\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 3 & 1 \\ 1 & 4 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.
$$

- You should be able to find the matrix of a linear transformation column by column. An example problem might ask to find the matrix for the linear transformation given by

$$
\begin{aligned}
y_1 &= 3x_1 + 2x_2 - x_3 \\
y_2 &= x_1 - 3x_2 + 2x_3.
\end{aligned}
$$

- You should be able to determine whether or not a transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear.

- You should be able to use rotation and rotation-dilation matrices:

$$\begin{pmatrix} a & -b \\ b & -a \end{pmatrix} = \begin{pmatrix} r\cos\theta & -r\sin\theta \\ r\sin\theta & r\cos\theta \end{pmatrix} = r \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

- You should be able to apply the definitions of shears, projections, and reflections.

Invertible Linear Transformations and Matrices

- You be able to determine whether or not a matrix (or a linear transformation) is invertible and be able to find the inverse if it exists.

Matrix Multiplication

- You should be able to interpret matrix multiplication in terms of linear transformations.

- You should be able to make the connections between the different representations of matrix multiplication.

- You should be able to compute matrix products column by column or entry by entry.

- You should be able to apply the rules of matrix algebra.

Chapter 3. Subspaces of $\mathbb{R}^n$ and Their Dimensions

- You should understand and be able to apply the concepts of the image and kernel of a linear transformation and be able to express the image and kernel of a matrix as the span of some vectors.

- You should be able to use the kernel and image of a matrix to determine whether or not the matrix is invertible.

- You should be able to verify whether or not a subset of $\mathbb{R}^n$ is a subspace.

- You should be able to understand and apply the concept of linear independence.

- You should be able to understand and apply the concept of a basis.

- You should understand and be able to use the concept of dimension.

- You should be able use the rank-nullity theorem for an $m \times n$ matrix.

- You should understand and be able to apply coordinate of a vector with respect to a particular basis and how to change coordinates.

Chapter 4. Introduction to Linear Spaces.

Linear Spaces

- You should understand and be able to apply the axioms of linear spaces. The important examples of linear spaces are:

  - The $m \times n$ matrices.
  - $C^\infty(\mathbb{R})$, the infinitely differentiable functions on $\mathbb{R}$.
  - $C[a, b]$, the continuous functions on $[a, b]$.
  - Polynomials.

  Don't forget the subspaces of each of these linear spaces.

- You should understand about span, linear independence, basis, and coordinates.

Linear Transformations and Isomorphisms

You should understand and be able to apply the definition of a linear transformation, $T : V \to W$.

- You should know that $\dim V = \operatorname{rank} T + \operatorname{nullity} T$ works for finite dimensional vector spaces but is not necessarily true for infinite dimensional linear spaces.

Isomorphisms

- If T is an isomorphism, the T^{-1} is an isomorphism.

- $T : V \to W$ is an isomorphism if and only if $\ker T = \{0\}$ and $\operatorname{im} T = W$.

- If $T : V \to W$ is an isomorphism and $f_1, \cdots, f_n$ is a basis of V, then $T(f_1), \ldots, T(f_n)$ is a basis for W.

- If $V \cong W$, then $\dim V = \dim W$.

Notes

- Omit Section 4.3. Coordinates in a Linear Space.

Chapter 5. Orthogonality and Least Squares.

Orthonormal Bases and Orthogonal Projections

- You should understand and be able to apply orthogonal and orthonormal sets.

- You should understand and be able to work with the orthogonal complement of a subspace V:

$$V^\perp = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{v} \cdot \mathbf{x} = \mathbf{0} \text{ for all } \mathbf{v} \in V\}.$$

  For example, you should be able to find the orthogonal complement of the image of

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 3 \\ 2 & 6 \end{pmatrix}$$

- You should understand and be able to work with orthogonal projections.

$$\text{proj}_V \mathbf{x} = (\mathbf{u}_1 \cdot \mathbf{x})\mathbf{u}_1 + \cdots + (\mathbf{u}_n \cdot \mathbf{x})\mathbf{u}_n$$

  for an orthonormal basis $\mathbf{u}_1, \cdots, \mathbf{u}_n$. You should know that proj_V is a linear map from $\mathbb{R}^n$ to V.

Gram-Schmidt Process and QR Factorization

- You should understand and be able to apply the Gram-Schmidt Process and QR Factorization.

Orthogonal Transformations and Orthogonal Matrices

You should understand and be able to apply facts about orthogonal matrices.

- Let A be an $n \times n$ matrix. The following statements are equivalent.

  1. The columns of the matrix A form an orthonormal set.
  2. $A^{-1} = A^T$.
  3. For vectors $\mathbf{x}$ and $\mathbf{y}$, $\langle A\mathbf{x}, A\mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle$.
  4. For vectors $\mathbf{x}$ and $\mathbf{y}$, $\|A\mathbf{x} - A\mathbf{y}\| = \|\mathbf{x} - \mathbf{y}\|$.
  5. For any vector $\mathbf{x}$, $\|A\mathbf{x}\| = \|\mathbf{x}\|$.

- You should understand and be able to apply the properties of the transpose of a matrix, including facts about symmetric and skew-symmetric matrices.

Least Squares and Data Fitting

- You should be able to find the least squares line $y = \alpha + \beta x$ that fits given data.

- You should understand and be able to solve the normal equations

$$A^T A \mathbf{x} = A^T \mathbf{b}.$$

Notes

- Omit Section 5.5. Inner Product Spaces.

Chapter 6. Determinants.

You should understand and be able to apply the basic properties of determinants.

- The determinant is a map $\det : A \mapsto \mathbb{R}$ such that an $n \times n$ matrix is invertible if and only if $\det(A) \neq 0$.

- $\det(AB) = \det(A) \det(B)$.

- If we interchange two columns of a matrix A to obtain a matrix A', then $-\det(A') = \det(A)$.

- If we multiply a column of a matrix A by α to obtain a matrix A', then $\det(A') = \alpha \det(A)$.

Remark. The determinant is only defined for square matrices.

- We have shown that the inverse of

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

  is

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

  It makes sense to try to define the determinant of A by

$$\det(A) = ad - bc.$$

  If we check, it satisfies all of other properties.

- **Sarrus's Rule.**

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}$$

- The determinant of a triangular matrix is the product of the diagonal entries.

Properties of Determinants

- $\det(A) = \det(A^T)$

- The determinant is linear in the ith coordinate. That is, if we define

$$T(\mathbf{x}) = (\mathbf{a}_1, \dots, \mathbf{a}_{i-1}, \mathbf{x}, \mathbf{a}_{i+1} \dots, \mathbf{a}_n)$$

then

$$T(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha T(\mathbf{x}) + \beta T(\mathbf{y}).$$

Since $\det(A) = \det(A^T)$, this also works for rows.

- If B is obtained from A by multiplying a row by $k \in \mathbb{R}$, then

$$\det(B) = k\det(A).$$

- If B is obtained from A by a single row swap, then

$$\det(B) = -\det(A).$$

- If B is obtained from A by adding a multiple of one row to another, then

$$\det(A) = \det(B).$$

- A is invertible if and only if $\det(A) \neq 0$.

- If A is invertible and we perform r row swaps and r divide various rows by s scalars to obtain I from A. Then

$$\det(A) = (-1)^s k_1 \cdots k_r.$$

- $\det(AB) = \det(A)\det(B)$.

- If A is an $n \times n$ invertible matrix, then

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

- If $B = S^{-1}AS$, then
$$\det(A) = \det(B).$$

Notes

- Omit Section 6.3. Geometric Interpretations of the Determinant.

Chapter 8. Eigenvalues, Eigenvectors and Eigenspaces

- You should understand and be able to apply the basic definitions of eigenvalue and eigenvector. An *eigenvector* for a linear transformation is one for which the transformation looks just like scalar multiplication. The associated *eigenvalue* is the scalar involved. You should be able to find all of the eigenvalues and eigenvectors for a given transformation.

- The eigenvalues of a matrix A are the roots of its characteristic polynomial. To find the eigenvectors of A, you need to find the kernel of the transformation $\lambda I - A$, where λ is an eigenvalue of A.

- You should understand the significance of and be able to find the algebraic and geometric multiplicity of an eigenvalue. You should understand and be able to apply the concept of an eigenbasis. An eigenbasis for a matrix A exists if and only if A is diagonalizable.

- You should understand the relationship between eigenvalues and similar matrices.

- You should have the ability to work with complex eigenvalues and eigenvectors.

- You should know Euler's formula (Fact 9.2.2).

- You should understand the definition of diagonalizable and be able to diagonalize a matrix.

- You should be able to use eigenvalues to solve problems involving discrete dynamical systems.

- You should understand and be able to apply the phase portraits of discrete and continuous dynamical systems in the case of 2×2 systems. See p. 420 for a nice summary.

- You should understand stability and know what happens in the long-term for specific cases. That is, zero is a stable equilibrium precisely when the modulus of all the eigenvalues is less than one.

Chapter 8. Symmetric Matrices

Section 8.1 is the only section covered from Chapter 8. You should understand and be able to apply the following the important facts.

- A symmetric matrix has real eigenvalues.

- **The Spectral Theorem.** A matrix is orthogonally diagonalizable if and only if it is symmetric.

Chapter 9. Continuous Dynamical Systems

- You should understand and be able to use the concept of a continuous dynamical system.

- You should be able to solve the differential equation,

$$\frac{dx}{dt} = kx.$$

- You should be able to solve the the system

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x},$$

  when A is diagonalizable over $\mathbb{R}$.

- You should be able to use $z = e^{\lambda t}$ to solve

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x},$$

  where $\lambda = a + bi$, where A is a 2×2 matrix.

- You should be able to use phase portraits to predict the behavior of solutions.

Section 9.3. Linear Differential Equations

- You should be able to solve second order linear differential equations of the form
$$ax'' + bx' + cx = 0.$$

- You may omit the operator approach to solving linear differential equations in section 9.3 (pp. 432–435)

Chapter 10.

- To understand the model for the heat equation.

- You should understand and be able to apply the separation of variables technique to find solutions to the heat equation.

- You should understand inner product spaces and be able to apply Fourier series to solve the heat equation.

Markov Chains

Please check with Prof. Ken Chung to see what you will be responsible for on the final exam.