Math 21b.
Review for Midterm I

Thomas W. Judson

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Chapter 1. Linear Equations

Solving Linear Systems

- To understand how systems of linear equations can be represented in a matrix.
- To understand and be able to use elementary row operations to compute the row echelon form and the reduced row echelon form of a matrix.
- To be able to use Gauss-Jordan elimination to solve linear systems.
- To be able to interpret solutions geometrically in $\mathbb{R}^2$ and $\mathbb{R}^3$.

Qualitative Facts about Solutions

- To be able to use the reduced row-echelon form of the augmented matrix to find the number of solutions of a linear system.
- To be able to apply the definition of the rank of a matrix.
- To understand that any underdetermined homogeneous system of linear equations must have a nontrivial solution.

Fact

Consider a system of m linear equations in n variables. Suppose the $m \times n$ coefficient matrix is A. Then

1. $\text{rank}(A) \leq m$ and $\text{rank}(A) \leq n$.
2. If $\text{rank}(A) = m$, then the system is consistent.
3. If $\text{rank}(A) = n$, then the system has at most one solution.
4. *Undetermined Systems.* if $\text{rank}(A) < n$, then the system has either no solution or an infinite number of solutions.
5. A linear system of n equations in n unknowns has a unique solution if and only if $\text{rank}(A) = n$.

Example Problems

- For which values of α does the following system have an infinite number of solutions?

$$\begin{aligned} x + 2y + z &= 0 \\ -x - y + z &= 0 \\ 3x + 4y + \alpha z &= 0 \end{aligned}$$

- Consider the following system of equations

$$\begin{aligned} x + 3y - z &= \beta_1 \\ x + 2y \quad &= \beta_2 \\ 3x + 7y - z &= \beta_3. \end{aligned}$$

  For which values of $\beta_1, \beta_2, \beta_3$ will the system be inconsistent?

Representing Linear Systems and Their Solutions

- To be able to compute the product of $A\mathbf{x}$ in terms of the columns or rows of A.
- To be able represent a linear system in vector or matrix form.

- To be able to represent all nontrivial solutions of a homogeneous system as a linear combination of vectors. For example, the solution of the system

$$\begin{aligned} 3x_1 + 2x_2 - x_3 - x_4 &= 0 \\ 2x_1 + 3x_2 + 6x_3 + 2x_4 &= 0 \\ x_1 - x_2 - 7x_3 - 3x_4 &= 0 \end{aligned}$$

  is $x_1 = 3s + (7/5)t$, $x_2 = -4s - (8/5)t$, $x_3 = s$, and $x_4 = t$. We can also write the solution as a linear combination of vectors,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = s \begin{pmatrix} 1 \\ -4 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 7/5 \\ -8/5 \\ 0 \\ 1 \end{pmatrix},$$

  called the *vector form of the general solution*.

Chapter 2. Linear Transformations

Linear Transformations

- To be able to use the concept of a linear transformation.

  A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is a *linear transformation* or *linear map* if there exists and $m \times n$ matrix A such that

$$T : \mathbf{x} \mapsto A\mathbf{x}$$

  for all $\mathbf{x} \in \mathbb{R}^n$.

  Equivalently, T satisfies the properties

$$\begin{aligned} T(\mathbf{x} + \mathbf{y}) &= T(\mathbf{x}) + T(\mathbf{y}); \\ T(\alpha\mathbf{x}) &= \alpha T(\mathbf{x}). \end{aligned}$$

- To be able to interpret simple linear transformations geometrically. For example, we may ask what happens to the unit square under the following transformations:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 3 & 1 \\ 1 & 4 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

- To be able to find the matrix of a linear transformation column by column. An example problem might ask to find the matrix for the linear transformation given by

$$\begin{aligned} y_1 &= 3x_1 + 2x_2 - x_3 \\ y_2 &= x_1 - 3x_2 + 2x_3. \end{aligned}$$

- To be able to determine whether or not a transformation is linear.
- To be able to use rotation and rotation-dilation matrices:

$$\begin{pmatrix} a & -b \\ b & -a \end{pmatrix} = \begin{pmatrix} r\cos\theta & -r\sin\theta \\ r\sin\theta & r\cos\theta \end{pmatrix} = r\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

- To be able to apply the definitions of shears, projections, and reflections.

Invertible Linear Transformations and Matrices

- To be able to determine whether or not a matrix (or a linear transformation) is invertible and to be able to find the inverse if it exists.
- **Example Problem.**

  For which values of α is the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 2 & \alpha \end{pmatrix}$$

  invertible? Find A^{-1}.

Matrix Multiplication

- To be able to interpret matrix multiplication in terms of linear transformations.
- To make the connections between the different representations of matrix multiplication.
- To be able to compute matrix products column by column or entry by entry.

- To be able to use the rules of matrix algebra.
- Typical problems.
  - True or False: The only 2×2 matrix A such that $A^2 = I_2$ is the identity matrix.
  - True or False: If $A^2 = A$, then $A = I$ or A is the zero matrix.
  - True or False: If A^2 is the zero matrix, then A must be the zero matrix.
  - True or False: If B has a column of zeros, then AB must have a column of zeros.

Chapter 3. Subspaces of $\mathbb{R}^n$ and Their Dimensions

- To understand and be able to apply the concepts of the image and kernel of a linear transformation and be able to express the image and kernel of a matrix as the span of some vectors.
- To be able to use the kernel and image of a matrix to determine whether or not the matrix is invertible.
- To be able to verify whether or not a subset of $\mathbb{R}^n$ is a subspace.
- To be able to understand and apply the concept of linear independence.
- **Example Problem.**

  Let A be an $n \times n$ invertible matrix and suppose that $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are linearly independent. If $\mathbf{y}_i = A\mathbf{x}_i$ for $i = 1, \ldots, n$, show that the vectors $\mathbf{y}_1, \ldots, \mathbf{y}_n$ are linearly independent.
- To be able to understand and apply the concept of a basis.
- **Example Problem.**

  If

$$A = \begin{pmatrix} 1 & 3 & 2 & 6 \\ 1 & 1 & 0 & 2 \\ 3 & 4 & 1 & 8 \\ 1 & 3 & 2 & 6 \end{pmatrix},$$

find a basis for $\ker(A)$ and $\text{image}(A)$.

- To understand and be able to use the concept of dimension.
- To be able use the rank-nullity theorem for an $m \times n$ matrix A:

$$\text{rank}(A) + \text{nullity}(A) = n.$$

- **Example Problem.** If A is a 5×4 matrix and the rank of A is two, find the dimension of the kernel of A.

A Word on Proofs

Let A be an $n \times n$ invertible matrix and B be an $n \times r$ matrix. Show that $\ker(AB) = \ker(B)$.