Transition Matrices: An Example

There are 10,000 commuters in a certain community.

- The number of people who commute by automobile is 8,000.
- The remaining 2,000 use public transportation.
- Every year 20% of the people who use mass transit change to commuting by automobile.
- Every year 30% of the people who drive to work change to using mass transit.

Assuming the total population of commuters remains constant, can we calculate how many people will use public transportation after 1 year, after 2 years, after n years? After 1 year the number of commuters who use automobiles and mass transit is

$$\begin{pmatrix} 0.70 & 0.20 \\ 0.30 & 0.80 \end{pmatrix} \begin{pmatrix} 8000 \\ 2000 \end{pmatrix}$$
$$= \begin{pmatrix} 0.70(8000) + 0.20(2000) \\ 0.30(8000) + 0.80(2000) \end{pmatrix}$$
$$= \begin{pmatrix} 6000 \\ 4000 \end{pmatrix}.$$

The matrix

$$A = \begin{pmatrix} 0.70 & 0.20 \\ 0.30 & 0.80 \end{pmatrix}$$

is similar to the matrix

$$B = \begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix}.$$

Observe

$$A = S^{-1}BS = \begin{pmatrix} -1 & 2/3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -3/5 & 2/5 \\ 3/5 & 3/5 \end{pmatrix}.$$

After n years, the number is given by

$$\begin{aligned} A^n \mathbf{x}_0 &= (S^{-1}BS)^n \mathbf{x}_0 \\ &= S^{-1}B^n S \mathbf{x}_0 \\ &= \begin{pmatrix} -1 & 2/3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} (1/2)^n & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -3/5 & 2/5 \\ 3/5 & 3/5 \end{pmatrix} \begin{pmatrix} 8000 \\ 2000 \end{pmatrix} \\ &= \begin{pmatrix} 4000(1/2)^n + 4000 \\ -4000(1/2)^n + 6000 \end{pmatrix} \\ &\to \begin{pmatrix} 4000 \\ 6000 \end{pmatrix}. \end{aligned}$$