Math 21b midsemester exam guide

Bio/statistics sections

a) The exam

- ☐ The midsemester exam is on *Tuesday March 23* from *7-9pm* in *Science Center Lecture Halls A and D*. Go to either hall.
- ☐ Here is a facsimile of the first page of the exam booklet:

MATH 21b Midsemester Exam
S 2004
Bio/statistics sections

1) ___ 2) ___ 3) ___ 4) ___ 5) ___ Total ______

Name: ____

Instructions:
- This exam booklet is only for students in the TTh 11:30-1 Bio/statistics section.
- Print your name in the line above and circle the time of your section.
- Answer each of the questions below in the space provided. If more space is needed, use the back of the facing page or on the extra blank pages at the end of this booklet. Please direct the grader to the extra pages used.
- Please give justification for answers if you are not told otherwise.
- Please write neatly. Answers deemed illegible by the grader will not receive credit.
- No calculators, computers or other electronic aids are allowed; nor are you allowed to refer to any written notes or source material; nor are you allowed to communicate with other students. Use only your brain and a pencil.
- You have 1 and 1/2 hours to complete your work.

In agreeing to take this exam, you are implicitly agreeing to act with fairness and honesty.

- ☐ As you can see, there are five problems. The first consists of a series of true/false questions, while the remaining four are more or less of the sort that you have been doing in your homework assignments.
- ☐ The exam will cover the material in Chapters 1-3 in the text book, Linear Algebra and Applications, in Chapters 1 and 3 of Probability and Statistics, and in Handouts 1-8.
- ☐ Advice for studying the linear algebra: There are plenty of answered problems in the text book and I strongly suggest that you work as many of these as you think necessary.

- ☐ Advice for studying the probability and statistics: Go over the material in the handouts and rework the homework problems to understand the concepts. Try to make up your own examples for the concepts that are mentioned below in Part c of this study guide. Not that the probability/statistics questions on the exam focus on the applications of these concepts in simple situations. Part d of this study guide contains some exercises (with answers) that you can use to test your conceptual understanding.
- ☐ Old exams: Old exams will not be terribly useful for the two reasons. First, in previous semesters, there were two midterm exams, not one. Thus, the exams from previous semesters will either test on material that we have not taught yet, or not test material that we have. Second, this version of Math 21b is very different from those taught in previous semesters.
- ☐ Of the topics covered so far, some I see as more important than others. Given below is a list of those that are especially important.

b) Linear algebra topics and issues to focus on

- ☐ Be able to the matrix that corresponds to a linear system of equations.
- ☐ Be able to rref(A) given the matrix A.
- ☐ Be able to solve $A\vec{x} = \vec{y}$ by computing rref for the augmented matrix, thus rref($A|\vec{y}$).
- ☐ Be able to find the inverse of a square matrix A by doing rref(A|I) where I is the identity matrix.
- ☐ Be able to use rref(A) to determine whether A is invertible, or if not, what its kernel is and what its image dimension is.
- ☐ Become comfortable with the notions that underlie the formal definitions of the following terms: linear transformation, linear subspace, the span of a set of vectors, linear dependence and linear independence, invertibility, orthogonality, kernel, image.
- ☐ Given a set, $\{\vec{v}_1, \ldots, \vec{v}_k\}$, of vectors in $\mathbb{R}^n$, be able to use the rref of the n-row/k-column matrix whose j'th column is $\vec{v}_j$ to determine if this set is linearly independent.
- ☐ Be able to find a basis for the kernel of a linear transformation.
- ☐ Be able to find a basis for the image of linear transformation.
- ☐ Know how to multiply matrices and also matrices against vectors. Know how these concepts respectively relate to the composition of two linear transformations and the action of a linear transformation.
- ☐ Know how the kernel and image of the product, AB, of matrices A and B are related to those of A and B.
- ☐ Understand how rref(AB) relates to rref(A) and rref(B).
- ☐ Be able to find the coordinates of a vector with respect to any given basis of $\mathbb{R}^n$.

- ☐ Be able to find the matrix of a linear transformation of $\mathbb{R}^n$ with respect to any given basis.

c) Probability topics and issues to focus on

- ☐ Be comfortable with the notions that underlie the terms sample space, event, probability function, conditional probability, random variable.
- ☐ Understand the intuition behind the properties in Equation (1.3) of Handout 1.
- ☐ Understand how to compute conditional probabilities.
- ☐ Understand the intuition behind Equation (1.5) of Handout 1. Understand how to interpret this equation as a linear equation. In some sense, this is the most important equation of all. Thus, you really need to know what it says and the conditions under which it applies.
- ☐ Know how to recognize when two events are independent.
- ☐ Know how to compute the conditional probability of an event A given B in terms of the the probabilities of A and B and the conditional probability of B given A. (Bayes' theorem.)
- ☐ Understand how to compute the probabilities for the values of a random variable. You should be comfortable with the fact that these probabilities define a probability function on the space of possible values of the random variable.
- ☐ Be able to compute the mean and standard deviation of a random variable given the probabilities of its values.
- ☐ Understand the definition of the correlation matrix for a pair of random variables, and understand its relation to the question of whether two random variables are independent.
- ☐ Understand the relationship between (4.7) and (4.6) in Handout 4.
- ☐ Understand the derivation of Equations (5.1) and (5.2) in Handout 5.
- ☐ Understand the derivation and meaning of Equations (5.5) and (5.6) in Handout 5.

d) Probability review exercises

These exercises are designed to cover many of the key ideas on probability that have been introduced so far. They will give you a rough idea of the probability issues that will appear on the mid-semester exam. Some are more involved and some less involved than the exam problems. The answers are at the end.

1. Suppose a certain quantity can take one of 4 values. Some N versions are observed. Give a sample space, S, for the suite of values that are possible for the collection of N measurements.

2. Let S denote the set of {0, 1, ..., N}. If k is in this set of integers, define P(k) to be $Z(\frac{1}{2})^k$. What must Z be so that P is a probability function?

3. Let S denote the set {1, 2, 3}. Let P denote a probability function on S that assigns non-zero probability to each element Can the sets {1, 2} and {2, 3} be independent?

4. Let S denote the set of pairs (β_1, β_2) where each $\beta_k \in \{1, 2, 3, 4\}$. Suppose that we set $P((\beta_1, \beta_2)) = \frac{1}{16}$. What is the conditional probability that $\beta_1+\beta_2 = 5$ given that $\beta_1 < 3$? What is the conditional probability that $\beta_1 < 3$ given that $\beta_1+\beta_2 = 5$?

5. Suppose that after each unit of time, one of two kinds of molecule is attached to the end of a growing chain. Label these molecules as α and β. Suppose that the probability of attaching α at time t is $\frac{1}{3}$ if the last molecule attached is α, and the probability of attaching β at time t is $\frac{2}{3}$ if the last molecule attached is α. Meanwhile, the probability of attaching β at time t is $\frac{1}{3}$ if the last molecule attached is β, and the probability of attaching α at time t is $\frac{2}{3}$ if the last molecule attached is β.
   a) What is the conditional probability of attaching α if the last molecule attached is α?
   b) What is the conditional probability of attaching α if the last molecule attached is β?
   c) Let p(t) denote the probability that the molecule attached at time t is α. Is it correct
      to say that $p(t) = \frac{2}{3} - \frac{1}{3}p(t-1)$?

6. Suppose that 20% of math professors have bad breath, 6% of the Science Center population has bad breath, and 1% of the Science Center population are math professors. You walk past someone on your way to class and their breath stuns you. What is the probability that this person is a math professor?

7. Let S denote the set of pairs (β_1, β_2) where each $\beta_k \in \{1, 2\}$. Let T = {1, 2, 4} denote the possible values of $\beta_1\beta_2$. Let Q denote the probability function on T that assigns $\frac{1}{2}$ to {4} and $\frac{1}{4}$ to both {1} and {2} Suppose that Q comes from a probability function, P, on S by viewing T as the possible values of the random variable that sends (β_1, β_2) to $\beta_1\beta_2$. Write down 3 linear equations with non-zero coefficients that are obeyed by the four values of P.

8. Let S and T denote the same set as in the previous problem. Let Q also be as in the previous problem and let P denote some probability function on S that gives Q for the

probabilities of the values of the random variable $(\beta_1, \beta_2) \to \beta_1\beta_2$. Compute the mean and standard deviation as determined by P for this random variable.

9. Let S, T, and P be as in the previous problem. Is the random variable on S that maps (β_1, β_2) to $\beta_1+\beta_2$ independent from the random variable that assigns $\beta_1\beta_2$?

10. Let {1, 2, 3, 4} label four kinds of small molecules that are attached end to end so as to make a growing chain. Suppose that molecule k is attached at the t'th stage of the chain with probability $\frac{3}{4}$ if the t-1'st molecule is of type k. If the t-1'st molecule is any given $j \neq k$, then the k'th molecule is attached with probability $\frac{1}{12}$. Let $\dot{p}(t)$ denote the 4-component vector whose k'th entry is the probability that the molecule of type k appears at the t'th stage of the chain. Find a 4×4 matrix, A, (not identically zero) that makes the equation $\dot{p}(t) = A\dot{p}(t-1)$ hold.

Answers

1. Once the 4-values are labeled by the integers from 1 through 4, the sample space consists of the 4^N possible N-tuples of the form $(\beta_1, \ldots, \beta_N)$ where each β_k can be either 1, 2, 3, or 4.

2. The constraint is that the sum of the probabilities must equal 1, so $Z = \frac{2^N}{2^{N+1}-1}$.

3. Never. Let A = {1, 2} and B = {2, 3}. Then $P(A \cap B) = P(2)$ and thus, if A and B are independent, then this must equal P(A)P(B). This is [P(1) + P(2)][P(2) + P(3)] = [1 – P(3)] [1 – P(1)] = 1 – P(1) – P(3) + P(1) P(3) = P(2) + P(1) P(3). This equals P(2) only if P(1) is zero or P(3) is zero.

4. The set, B, where $\beta_1 < 3$ has eight elements so has probability $\frac{1}{2}$. The set, A, where the sum is 5 has 4 elements so has probability $\frac{1}{4}$. The intersection, $A \cap B$, this the set where $\beta_1 < 3$ and $\beta_1+\beta_2 = 5$, consists of two elements and has probability $\frac{1}{8}$. Then, the conditional probability $P(A|B) = \frac{1}{8}/\frac{1}{2} = \frac{1}{4}$. The conditional probability P(B|A) is the ratio $\frac{1}{8}/\frac{1}{4} = \frac{1}{2}$.

5. a) You are told that this is $\frac{1}{3}$.
   b) You are told that this is $\frac{2}{3}$.
   c) Use the formula $p(t) = \text{Prob}(\alpha \text{ at } t|\alpha \text{ at } t\text{-}1)\, p(t\text{-}1) + \text{Prob}(\alpha \text{ at } t|\beta \text{ at } t\text{-}1)\, p(\beta \text{ at } t\text{-}1)$. Since $p(\beta \text{ at } t\text{-}1) = (1 - p(t))$, this says that $p(t) = \frac{1}{3}p(t\text{-}1) + \frac{2}{3}(1\text{-}p(t)) = \frac{2}{3} - \frac{1}{3}p(t\text{-}1)$.

6. This is an exercise using Bayes' theorem. Let A = event the person is a math professor, and B = event the person has bad breath. You are asked to find P(A|B) and you are given P(A) = 0.01, P(B) = 0.06 and P(B|A) = 0.2. Since P(A|B) = P(B|A)·P(A)/P(B), this gives P(A|B) = 0.0333···.

7. $Q(1) = \frac{1}{4} = P(1,1)$.
   $Q(2) = \frac{1}{4} = P(1,2) + P(2,1)$.
   $Q(4) = \frac{1}{2} = P(2,2)$.

8. The mean is $1\cdot\frac{1}{4} + 2\cdot\frac{1}{4} + 4\cdot\frac{1}{2} = \frac{11}{4}$. The standard deviation is the square root of the following sum: $\frac{49}{16}\cdot\frac{1}{4} + \frac{9}{16}\cdot\frac{1}{4} + \frac{25}{16}\cdot\frac{1}{2} = \frac{27}{16}$. Thus, $\sigma = \frac{3}{4}\sqrt{3}$.

9. The random variables are not independent since you can determine the value of one knowing that of the other. Listing the values of $\beta_1+\beta_2$ as {2, 3, 4} and those of $\beta_1\beta_2$ as {1, 2, 4}, the corresponding 3×3 correlation matix is

$$\begin{pmatrix} \frac{3}{16} & -\frac{1}{16} & -\frac{1}{8} \\ -\frac{1}{16} & \frac{3}{16} & -\frac{1}{8} \\ -\frac{1}{8} & -\frac{1}{8} & \frac{1}{4} \end{pmatrix}$$

10. Let $p_k(t)$ denote the k'th entry of $\dot{p}(t)$. According to the scenario given,

$$A = \begin{pmatrix} \frac{3}{4} & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} \\ \frac{1}{12} & \frac{3}{4} & \frac{1}{12} & \frac{1}{12} \\ \frac{1}{12} & \frac{1}{12} & \frac{3}{4} & \frac{1}{12} \\ \frac{1}{12} & \frac{1}{12} & \frac{1}{12} & \frac{3}{4} \end{pmatrix}.$$

Note that in this example, $p_k(t) = \frac{3}{4} p_k(t-1) + \frac{1}{4}(1-p_k(t-1)) = \frac{1}{4} + \frac{1}{2} p_k(t-1)$.