MATRIX PRODUCT	Math 21b, O. Knill

MATRIX PRODUCT. If B is a $p \times m$ matrix and A is a $m \times n$ matrix, then BA is defined as the $p \times n$ matrix with entries $(BA)_{ij} = \sum_{k=1}^{m} B_{ik} A_{kj}$.

EXAMPLE. If B is a 3×4 matrix, and A is a 4×2 matrix then BA is a 3×2 matrix.

$$B = \begin{bmatrix} 1 & 3 & 5 & 7 \\ 3 & 1 & 8 & 1 \\ 1 & 0 & 9 & 2 \end{bmatrix}, A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, BA = \begin{bmatrix} 1 & 3 & 5 & 7 \\ 3 & 1 & 8 & 1 \\ 1 & 0 & 9 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 3 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 15 & 13 \\ 14 & 11 \\ 10 & 5 \end{bmatrix}.$$

COMPOSING LINEAR TRANSFORMATIONS. If $S : \mathbf{R}^n \to \mathbf{R}^m, x \mapsto Ax$ and $T : \mathbf{R}^m \to \mathbf{R}^p, y \mapsto By$ are linear transformations, then their composition $T \circ S : x \mapsto BA(x))$ is a linear transformation from $\mathbf{R}^n$ to $\mathbf{R}^p$. The corresponding matrix is the matrix product $B \cdot A$.

EXAMPLE. Find the matrix which is a composition of a rotation around the x-axes by an agle $\pi/2$ followed by a rotation around the z-axes by an angle $\pi/2$.
SOLUTION. The first transformation has the property that $e_1 \to e_1, e_2 \to e_3, e_3 \to -e_2$, the second $e_1 \to e_2, e_2 \to -e_1, e_3 \to e_3$. If A is the matrix belonging to the first transformation and B the second, then BA is the matrix to the composition. The composition maps $e_1 \to -e_2 \to e_3 \to e_1$ is a rotation around a long diagonal. $B = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$, $BA = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

EXAMPLE. A rotation dilation is the composition of a rotation by $\alpha = \arctan(b/a)$ and a dilation (=scale) by $r = \sqrt{a^2 + b^2}$.

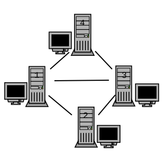
REMARK. Matrix multiplication is a generalization of usual multiplication of numbers or the dot product.

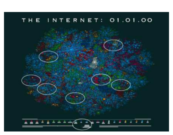
MATRIX ALGEBRA. Note that $AB \neq BA$ in general! Otherwise, the same rules apply as for numbers: $A(BC) = (AB)C$, $AA^{-1} = A^{-1}A = 1_n$, $(AB)^{-1} = B^{-1}A^{-1}$, $A(B + C) = AB + AC$, $(B + C)A = BA + CA$ etc.

PARTITIONED MATRICES. The entries of matrices can themselves be matrices. If B is a $m \times n$ matrix and A is a $n \times p$ matrix, and assume the entries are $k \times k$ matrices, then BA is a $m \times p$ matrix where each entry $(BA)_{ij} = \sum_{k=1}^{n} B_{ik} A_{kj}$ is a $k \times k$ matrix. Partitioning matrices can be useful to improve the speed of matrix multiplication (i.e. Strassen algorithm).

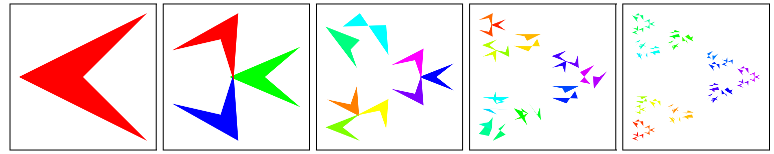
EXAMPLE. If $A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$, where A_{ij} are $k \times k$ matrices with the property that A_{11} and A_{22} are invertible, then $B = \begin{bmatrix} A_{11}^{-1} & -A_{11}^{-1} A_{12} A_{22}^{-1} \\ 0 & A_{22}^{-1} \end{bmatrix}$ is the inverse of A.

APPLICATIONS. (The material which follows is for motivation puposes only, more applications appear in the homework).



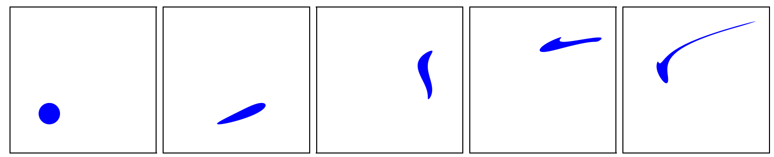


NETWORKS. Let us associate to the computer network (shown at the left) a matrix $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$ To a worm in the first computer we associate a vector $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. The vector Ax has a 1 at the places, where the worm could be in the next step. The vector $(AA)(x)$ tells, in how many ways the worm can go from the first computer to other hosts in 2 steps. In our case, it can go in three different ways back to the computer itself.
Matrices help to solve combinatorial problems (see movie "Good will hunting"). For example, what does $[A^{1000}]_{22}$ tell about the worm infection of the network? What does it mean if A^{100} has no zero entries?



FRACTALS. Closely related to linear maps are **affine maps** $x \mapsto Ax + b$. They are compositions of a linear map with a translation. It is **not** a linear map if $B(0) \neq 0$. Affine maps can be disguised as linear maps in the following way: let $y = \begin{bmatrix} x \\ 1 \end{bmatrix}$ and defne the $(n+1)*(n+1)$ matrix $B = \begin{bmatrix} A & b \\ 0 & 1 \end{bmatrix}$. Then $By = \begin{bmatrix} Ax + b \\ 1 \end{bmatrix}$.

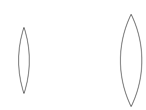
Fractals can be constructed by taking for example 3 affine maps R, S, T which contract area. For a given object Y_0 define $Y_1 = R(Y_0) \cup S(Y_0) \cup T(Y_0)$ and recursively $Y_k = R(Y_{k-1}) \cup S(Y_{k-1}) \cup T(Y_{k-1})$. The above picture shows Y_k after some iterations. In the limit, for example if $R(Y_0), S(Y_0)$ and $T(Y_0)$ are disjoint, the sets Y_k converge to a **fractal**, an object with dimension strictly between 1 and 2.



CHAOS. Consider a map in the plane like $T : \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 2x + 2\sin(x) - y \\ x \end{bmatrix}$ We apply this map again and again and follow the points $(x_1, y_1) = T(x, y)$, $(x_2, y_2) = T(T(x, y))$, etc. One writes T^n for the n-th iteration of the map and (x_n, y_n) for the image of (x, y) under the map T^n. The linear approximation of the map at a point (x, y) is the matrix $DT(x, y) = \begin{bmatrix} 2 + 2\cos(x) - 1 \\ 1 \end{bmatrix}$. (If $T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f(x, y) \\ g(x, y) \end{bmatrix}$, then the row vectors of $DT(x, y)$ are just the gradients of f and g). T is called **chaotic at** (x, y), if the entries of $D(T^n)(x, y)$ grow exponentially fast with n. By the **chain rule**, $D(T^n)$ is the product of matrices $DT(x_i, y_i)$. For example, T is chaotic at $(0, 0)$. If there is a positive probability to hit a chaotic point, then T is called chaotic.

FALSE COLORS. Any color can be represented as a vector (r, g, b), where $r \in [0, 1]$ is the red $g \in [0, 1]$ is the green and $b \in [0, 1]$ is the blue component. Changing colors in a picture means applying a transformation on the cube. Let $T : (r, g, b) \mapsto (g, b, r)$ and $S : (r, g, b) \mapsto (r, g, 0)$. What is the composition of these two linear maps?

OPTICS. Matrices help to calculate the motion of light rays through lenses. A light ray $y(s) = x + ms$ in the plane is described by a vector (x, m). Following the light ray over a distance of length L corresponds to the map $(x, m) \mapsto (x + mL, m)$. In the lens, the ray is bent depending on the height x. The transformation in the lens is $(x, m) \mapsto (x, m - kx)$, where k is the strength of the lense.



$$\begin{bmatrix} x \\ m \end{bmatrix} \mapsto A_L \begin{bmatrix} x \\ m \end{bmatrix} = \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ m \end{bmatrix}, \begin{bmatrix} x \\ m \end{bmatrix} \mapsto B_k \begin{bmatrix} x \\ m \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -k & 1 \end{bmatrix} \begin{bmatrix} x \\ m \end{bmatrix}.$$

Examples:
1) Eye looking far: $A_R B_k$. 2) Eye looking at distance L: $A_R B_k A_L$.
3) Telescope: $B_{k_2} A_L B_{k_1}$. (More about it in problem 80 in section 2.4).