24. a. $\langle f, g+h\rangle = \langle f, g\rangle + \langle f, h\rangle = 0+8=8$

b. $\|g+h\| = \sqrt{\langle g+h, g+h\rangle} = \sqrt{\langle g,g\rangle + 2\langle g,h\rangle + \langle h,h\rangle} = \sqrt{1+6+50} = \sqrt{57}$

c. Since $\langle f, g\rangle = 0$, $\|g\| = 1$, and $\|f\| = 2$, we know that $\frac{f}{2}, g$ is an orthonormal basis of span $(f,\ g)$.

Now $\text{proj}_E h = \left\langle \frac{f}{2}, h\right\rangle \frac{f}{2} + \langle g,h\rangle g = \frac{1}{4}\langle f,h\rangle f + \langle g,h\rangle g = 2f+3g.$

d. From part c we know that $\frac{1}{2}f,\ g$ are orthonormal, so we apply Algorithm 5.2.1 to obtain the third polynomial in an orthonormal basis of span$(f,\ g,\ h)$:

$$\frac{h - \text{proj}_E h}{\|h - \text{proj}_E h\|} = \frac{h-2f-3g}{\|h-2f-3g\|} = \frac{h-2f-3g}{5} = -\frac{2}{5}f - \frac{3}{5}g + \frac{1}{5}h$$

Orthonormal basis: $\frac{1}{2}f, g, -\frac{2}{5}f - \frac{3}{5}g + \frac{1}{5}h$

25. Using the inner product defined in Example 2, we find that

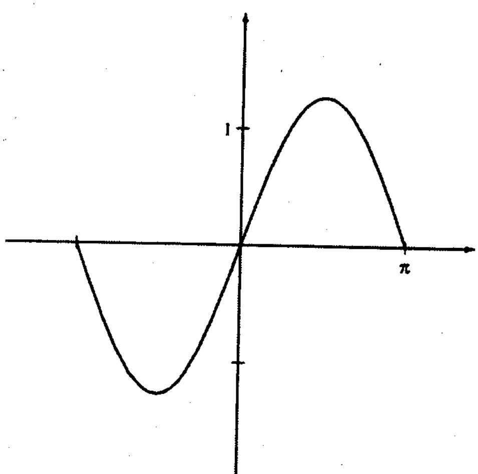
$$\|\vec{x}\| = \sqrt{\langle \vec{x}, \vec{x}\rangle} = \sqrt{1+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{n^2}+\cdots} = \sqrt{\frac{\pi^2}{6}} = \frac{\pi}{\sqrt{6}}$$ (see the text right after Fact 5.5.6).

26. $a_0 = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(t)\,dt = 0$

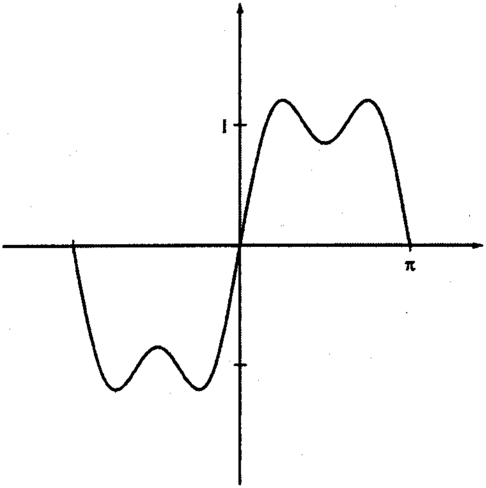
$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(kt)\,dt = \frac{1}{\pi}\left\{-\int_{-\pi}^{0}\sin(kt)\,dt + \int_{0}^{\pi}\sin(kt)\,dt\right\} = \frac{2}{\pi}\int_0^{\pi}\sin(kt)\,dt$$

$$= -\frac{2}{k\pi}[\cos(kt)]_0^{\pi} = \begin{cases} 0 & \text{if } k \text{ is even} \\ \frac{4}{\pi k} & \text{if } k \text{ is odd} \end{cases}$$

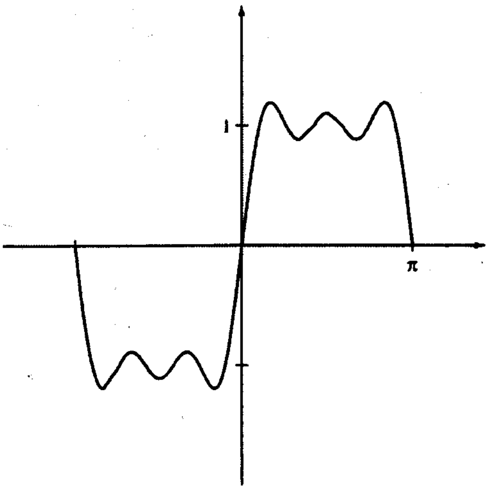
$c_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(kt)\,dt = 0$, since the integrand is odd.



$$f_1(t) = f_2(t) = \frac{4}{\pi}\sin(t)$$



$$f_3(t) = f_4(t) = \frac{4}{\pi}\sin(t) + \frac{4}{3\pi}\sin(3t)$$



$$f_5(t) = f_6(t) = \frac{4}{\pi}\sin(t) + \frac{4}{3\pi}\sin(3t) + \frac{4}{5\pi}\sin(5t)$$