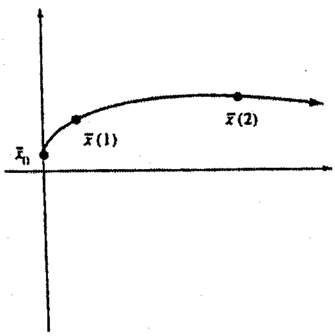
6. $\lambda_{1,2} = 0.8 \pm (0.6)i$ so $|\lambda_1| = |\lambda_2| = \sqrt{0.64 + 0.36} = 1$ and $\vec{0}$ is not a stable equilibrium.

16. $\lambda_{1,2} = \dfrac{2 \pm \sqrt{1+30k}}{10}$ so $|2 \pm \sqrt{1+30k}|$ must be less than 10. $\lambda_{1,2}$ are *real* if $k \geq -\frac{1}{30}$. In this case it is required that $2 + \sqrt{1+30k} < 10$ and $-10 < 2 - \sqrt{1+30k}$, which means that $\sqrt{1+30k} < 8$ or $k < \frac{21}{10}$. $\lambda_{1,2}$ are *complex* if $k < -\frac{1}{30}$. Here it is required that $4 + (-1 - 30k) < 100$ or $k > -\frac{97}{30}$. Overall, $\vec{0}$ is a stable equilibrium if $-\frac{97}{30} < k < \frac{21}{10}$.

20. $\lambda_{1,2} = 4 \pm 3i, r = 5, \phi = \arctan\left(\frac{3}{4}\right) \approx 0.64$, so $\lambda_1 \approx 5(\cos(0.64) + i\sin(0.64))$, $[\vec{w}\ \vec{v}] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{x}(t) \approx 5^t \begin{bmatrix} \sin(0.64t) \\ \cos(0.64t) \end{bmatrix}$.

Spirals outwards (rotation-dilation).



28. Not stable, since if λ is an eigenvalue of A, then $(\lambda - 2)$ is an eigenvalue of $(A - 2I_n)$ and $|\lambda - 2| > 1$.

38. a. $T(\vec{v}) = A\vec{v} + \vec{b} = \vec{v}$ if $\vec{v} - A\vec{v} = \vec{b}$ or $(I_n - A)\vec{v} = \vec{b}$.
$I_n - A$ is invertible since 1 is not an eigenvalue of A. Therefore, $\vec{v} = (I_n - A)^{-1}\vec{b}$ is the only solution.

b. Let $\vec{y}(t) = \vec{x}(t) - \vec{v}$ be the deviation of $\vec{x}(t)$ from the equilibrium $\vec{v}$.
Then $\vec{y}(t+1) = \vec{x}(t+1) - \vec{v} = A\vec{x}(t) + \vec{b} - \vec{v} = A(\vec{y}(t) + \vec{v}) + \vec{b} - \vec{v} = A\vec{y}(t) + A\vec{v} + \vec{b} - \vec{v} = A\vec{y}(t)$,
so that $\vec{y}(t) = A^t\vec{y}(0)$, or $\vec{x}(t) = \vec{v} + A^t(\vec{x}_0 - \vec{v})$.
$\lim_{t\to\infty} \vec{x}(t) = \vec{v}$ for all $\vec{x}_0$ if $\lim_{t\to\infty} A^t(\vec{x}_0 - \vec{v}) = \vec{0}$. This is the case if the modulus of all the eigenvalues of A is less than 1.

42. a. $x(t+1) = x(t) - ky(t)$
$y(t+1) = kx(t) + y(t) = kx(t) + (1-k^2)y(t)$ so $\begin{bmatrix} x(t+1) \\ y(t+1) \end{bmatrix} = \begin{bmatrix} 1 & -k \\ k & 1-k^2 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$.

b. $f_A(\lambda) = \lambda^2 - (2-k^2)\lambda + 1 = 0$
The discriminant is $(2-k^2)^2 - 4 = -4k^2 + k^4 = k^2(k^2-4)$, which is negative if k is a small positive number ($k < 2$). Therefore, the eigenvalues are complex. By Fact 7.6.4 the trajectory will be an ellipse, since $\det(A) = 1$.

8. $\frac{1}{\sqrt{10}}\begin{bmatrix}3\\1\end{bmatrix}, \frac{1}{\sqrt{10}}\begin{bmatrix}-1\\3\end{bmatrix}$ is an orthonormal eigenbasis, with $\lambda_1 = 4$ and $\lambda_2 = -6$, so $S = \frac{1}{\sqrt{10}}\begin{bmatrix}3 & -1\\1 & 3\end{bmatrix}$ and $D = \begin{bmatrix}4 & 0\\0 & -6\end{bmatrix}$.

9. $\frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}$ is an orthonormal eigenbasis, with $\lambda_1 = 3$, $\lambda_2 = -3$, and $\lambda_3 = 2$, so

$S = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1 & 0\\0 & 0 & \sqrt{2}\\1 & 1 & 0\end{bmatrix}$ and $D = \begin{bmatrix}3 & 0 & 0\\0 & -3 & 0\\0 & 0 & 2\end{bmatrix}$.

10. $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 = 9$.

$\vec{v}_1 = \frac{1}{\sqrt{5}}\begin{bmatrix}2\\1\\0\end{bmatrix}$ is in E_0 and $\vec{v}_2 = \frac{1}{3}\begin{bmatrix}1\\-2\\2\end{bmatrix}$ is in E_9.

Let $\vec{v}_3 = \vec{v}_1 \times \vec{v}_2 = \frac{1}{3\sqrt{5}}\begin{bmatrix}2\\-4\\-5\end{bmatrix}$; then $\vec{v}_1, \vec{v}_2, \vec{v}_3$ is an orthonormal eigenbasis.

16. a. $\ker(A)$ is four-dimensional, so that the eigenvalue 0 has multiplicity 4, and the remaining eigenvalue is $\mathrm{tr}(A) = 5$.

b. $B = A + 2I_5$, so that the eigenvalues are 2, 2, 2, 2, 7.

c. $\det(B) = 2^4 \cdot 7 = 112$ (product of eigenvalues)

20. By Exercise 19, there is an orthonormal basis $\vec{v}_1, \dots, \vec{v}_n$ of $\mathbb{R}^n$ such that $T(\vec{v}_1), \dots, T(\vec{v}_n)$ are orthogonal. Suppose that $T(\vec{v}_1), \dots, T(\vec{v}_r)$ are nonzero and $T(\vec{v}_{r+1}), \dots, T(\vec{v}_n)$ are zero. Then let $\vec{w}_i = \frac{1}{\|T(\vec{v}_i)\|}T(\vec{v}_i)$ for $i = 1, \dots, r$ and choose an orthonormal basis $\vec{w}_{r+1}, \dots, \vec{w}_m$ of $[\mathrm{span}(\vec{w}_1, \dots, \vec{w}_r)]^\perp$. Then $\vec{w}_1, \dots, \vec{w}_m$ does the job.

24. Note that A is symmetric and orthogonal, so that the eigenvalues are 1 and -1 (see Exercise 23).

$E_1 = \mathrm{span}\left(\begin{bmatrix}1\\0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\\0\end{bmatrix}\right)$ and $E_{-1} = \mathrm{span}\left(\begin{bmatrix}1\\0\\0\\-1\end{bmatrix}, \begin{bmatrix}0\\1\\-1\\0\end{bmatrix}\right)$, so that

$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\0\\1\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\1\\0\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\0\\-1\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\-1\\0\end{bmatrix}$ is an orthonormal eigenbasis.

30. The columns $\vec{v}, \vec{v}_2, \dots, \vec{v}_n$ of R form an orthogonal eigenbasis for $A = \vec{v}\vec{v}^T$, with eigenvalues $1, 0, 0, \dots, 0$ ($n-1$ zeros), since
$A\vec{v} = \vec{v}\vec{v}^T\vec{v} = \vec{v}(\vec{v}\cdot\vec{v}) = \vec{v}$, (since $\vec{v}\cdot\vec{v} = 1$) and $A\vec{v}_i = \vec{v}\vec{v}^T\vec{v}_i = \vec{v}(\vec{v}\cdot\vec{v}_i) = \vec{0}$ (since $\vec{v}\cdot\vec{v}_i = 0$).

Therefore we can let $S = R$, and $D = \begin{bmatrix}1 & 0 & \cdots & 0\\0 & 0 & \cdots & 0\\\vdots & & & \\0 & 0 & \cdots & 0\end{bmatrix}$.

36. If $\vec{v}$ is an eigenvector with eigenvalue λ, then $\lambda\vec{v} = A\vec{v} = A^2\vec{v} = \lambda^2\vec{v}$, so that $\lambda = \lambda^2$ and therefore $\lambda = 0$ or $\lambda = 1$. Since A is symmetric, E_0 and E_1 are orthogonal complements, so that A represents the orthogonal projection onto E_1.

14. a. $x(t) = e^{-\frac{t}{8270}}$, by Fact 9.1.1

If T is the half-life, then $e^{-\frac{T}{8270}} = \frac{1}{2}$ or $-\frac{T}{8270} = \ln\left(\frac{1}{2}\right)$ or $T = -8270\ln\left(\frac{1}{2}\right) \approx 5732$.

The half-life is about 5732 years.

b. We want to find t such that $e^{-\frac{t}{8270}} = 1 - 0.47 = 0.53$ or $-\frac{t}{8270} = \ln(0.53)$ or $t = -8270\ln(0.53) \approx$ 5250. The Iceman died about 5000 years before A.D. 1991, or about 3000 B.C. The Austrian expert was wrong.

22. We are told that $\frac{d\vec{x}_1}{dt} = A\vec{x}_1$ and $\frac{d\vec{x}_2}{dt} = A\vec{x}_2$. Let $\vec{x}(t) = \vec{x}_1(t) + \vec{x}_2(t)$. Then $\frac{d\vec{x}}{dt} = \frac{d\vec{x}_1}{dt} + \frac{d\vec{x}_2}{dt} = A\vec{x}_1 +$ $A\vec{x}_2 = A(\vec{x}_1 + \vec{x}_2) = A\vec{x}$, as claimed.

23. We are told that $\frac{d\vec{x}_1}{dt} = A\vec{x}_1$. Let $\vec{x}(t) = k\vec{x}_1(t)$. Then $\frac{d\vec{x}}{dt} = \frac{d}{dt}(k\vec{x}_1) = k\frac{d\vec{x}_1}{dt} = kA\vec{x}_1 = A(k\vec{x}_1) = A\vec{x}$, as claimed.

24. We are told that $\frac{d\vec{x}}{dt} = A\vec{x}$. Let $\vec{c}(t) = e^{kt}\vec{x}(t)$. Then $\frac{d\vec{c}}{dt} = \frac{d}{dt}(e^{kt}\vec{x}) = \left(\frac{d}{dt}e^{kt}\right)\vec{x} + e^{kt}\frac{d\vec{x}}{dt} = ke^{kt}\vec{x} +$ $e^{kt}A\vec{x} = (A + kI_n)(e^{kt}\vec{x}) = (A + kI_n)\vec{c}$, as claimed.

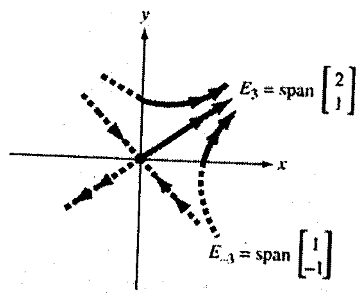
28. $\lambda_1 = 2, \lambda_2 = 10$; $\vec{v}_1 = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$; $c_1 = -\frac{1}{8}$, $c_2 = \frac{5}{8}$, so that $\vec{x}(t) = -\frac{1}{8}e^{2t}\begin{bmatrix} -3 \\ 2 \end{bmatrix} + \frac{5}{8}e^{10t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

40. $\vec{x}(t) = e^{2t}\begin{bmatrix} 2 \\ 3 \end{bmatrix} + e^{3t}\begin{bmatrix} 3 \\ 4 \end{bmatrix}$

We want a 2×2 matrix A with eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 3$ and associated eigenvectors $\vec{v}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$; that is $A\begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 9 \\ 6 & 12 \end{bmatrix}$ or $A = \begin{bmatrix} 4 & 9 \\ 6 & 12 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} 4 & 9 \\ 6 & 12 \end{bmatrix}\begin{bmatrix} -4 & 3 \\ 3 & -2 \end{bmatrix}$ $= \begin{bmatrix} 11 & -6 \\ 12 & -6 \end{bmatrix}$.

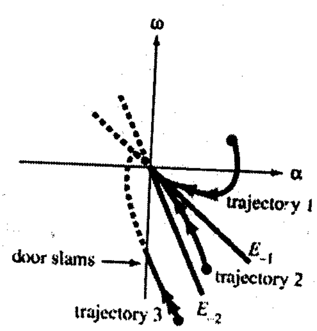
44. a. The two species are in symbiosis: Each is helped by the other (consider the terms $4y$ and $2x$).

b.



c. Both populations will prosper and $\lim_{t\to\infty} \frac{y(t)}{x(t)} = \frac{1}{2}$, regardless of the initial populations.

54. $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$, $\lambda_1 = -1, \lambda_2 = -2$; $\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.



In the case of trajectory 3 the door will slam: Initially the door is opened just a little (α is small) and given a strong push to close it (ω is large negative). More generally, the door will slam if the point $\begin{bmatrix} \alpha(0) \\ \omega(0) \end{bmatrix}$ representing the initial state is located below the line $E_{-2} = \text{span}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$, that is, if $\frac{\omega(0)}{\alpha(0)} < -2$.