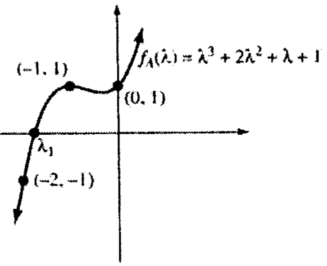
12. Yes, the zero state is stable. To see this, use technology to determine that the real parts of the eigenvalues are all negative. Or draw a rough sketch of the characteristic polynomial to see that there is one real eigenvalue λ_1 between -1 and -2; the two other eigenvalues must be $\lambda_{2,3} = p \pm iq$ with $p < 0$, since $\lambda_1 + \lambda_2 + \lambda_3 = \lambda_1 + 2p = \operatorname{tr}(A) = -2$.



13. Recall that the zero state is stable if (and only if) the real parts of all eigenvalues are negative. Now the eigenvalues of A^{-1} are the reciprocals of those of A; the real parts have the same sign $\left(\text{if } \lambda = p + iq, \text{ then } \frac{1}{\lambda} = \frac{1}{p+iq} = \frac{p-iq}{p^2+q^2}\right)$.

14. a. For $i > 1$, $\frac{dx_i}{dt} = -k_i x_i + x_{i-1}$. This means that in the absence of quantity $x_{i-1}(t)$, the quantity $x_i(t)$ will decay exponentially, but the presence of x_{i-1} helps x_i to grow.

For $i = 1$, the beginning of the loop, $\frac{dx_1}{dt} = -k_1 x_1 - b x_n$, so that the presence of x_n contributes to the decrease of x_1.

b. If $n = 2$ then the matrix of the system is $A = \begin{bmatrix} -k_1 & -b \\ 1 & -k_2 \end{bmatrix}$ with $\operatorname{tr}(A) = -k_1 - k_2 < 0$ and $\det(A) = k_1 k_2 + b > 0$, so that the zero state is stable, by Fact 9.2.5.

c. No: consider the case $k_1 = k_2 = k_3 = 1$ for simplicity; then the matrix of the system is $A = \begin{bmatrix} -1 & 0 & b \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$ and $f_A(\lambda) = (\lambda + 1)^3 - b$. If b exceeds 1, then the matrix A will have a positive eigenvalue, so that the zero state is not stable.

16. If $A = \begin{bmatrix} 0 & 1 \\ a & b \end{bmatrix}$ then $\operatorname{tr}(A) = b$ and $\det(A) = -a$. By Fact 9.2.5, the zero state is stable if a and b are both negative.

21. a. $\left| \begin{array}{lll} \frac{db}{dt} = 0.05b + & & s \\ \frac{ds}{dt} = & & 0.07s \end{array} \right|$ and $\begin{bmatrix} b(0) \\ s(0) \end{bmatrix} = \begin{bmatrix} 1000 \\ 1000 \end{bmatrix}$

b. $\lambda_1 = 0.07, \lambda_2 = 0.05$; $\vec{v}_1 = \begin{bmatrix} 50 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$; $\vec{x}(0) = 1000\vec{v}_1 - 49000\vec{v}_2$; so that

$b(t) = 50000e^{0.07t} - 49000e^{0.05t}$ and $s(t) = 1000e^{0.07t}$.

22. $\lambda_1 = 3, \lambda_2 = 0.5$; $E_3 = \operatorname{span}\begin{bmatrix} 1 \\ -1 \end{bmatrix}, E_{0.5} = \operatorname{span}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

System is discrete so choose VII.

23. $\lambda_{1,2} = -\frac{1}{2} \pm i, r > 1$, so that trajectory spirals outwards. Choose II.

24. $\lambda_1 = 3, \lambda_2 = 0.5, E_3 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, E_{0.5} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

System is continuous, so choose I.

26. $\lambda_1 = 1, \lambda_2 = -2$; $E_1 = \operatorname{span}\begin{bmatrix} 0 \\ 1 \end{bmatrix}, E_{-2} = \operatorname{span}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

System is continuous so choose V.

28. $\lambda_{1,2} = \pm 6i$, $E_{6i} = \text{span}\left(\begin{bmatrix}2\\0\end{bmatrix} + i\begin{bmatrix}0\\3\end{bmatrix}\right)$, so that

$$\vec{x}(t) = \begin{bmatrix}0 & 2\\3 & 0\end{bmatrix}\begin{bmatrix}\cos(6t) & -\sin(6t)\\ \sin(6t) & \cos(6t)\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix} = \begin{bmatrix}2\sin(6t) & 2\cos(6t)\\3\cos(6t) & -3\sin(6t)\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix}.$$

38. a. $\left(\frac{z_1}{z_2}\right)' = \frac{z'_1 z_2 - z_1 z'_2}{z_2^2} = \frac{\lambda z_1 z_2 - \lambda z_1 z_2}{z_2^2} = 0$, so that $\frac{z_1(t)}{z_2(t)} = k$, a constant.
Now $z_1(t) = k z_2(t)$; substituting $t = 0$ gives $1 = z_1(0) = k z_2(0) = k$, so that $z_1(t) = z_2(t)$, as claimed.

b. Let $z_2(t) = e^{pt}(\cos(qt) + i\sin(qt))$ be the solution constructed in the text (page 414). Since $z_2(t) \neq 0$ for all t, this is the only solution, by part a.

Section 10.1 Answers

Section 10.1:

1. b) and c).
2. a), c).
3. a), c), d).
7. A basis is $\{t, -2+t^2\}$.
8. A basis is $\{1+t^2, t+t^2\}$.

Section 9.4 Solutions

1. a. Solving for the nullclines, we get

$$\frac{dx}{dt} = x(2 - x + y) = 0 \Rightarrow \begin{cases} x = 0 & \text{or} \\ y = x - 2 & \end{cases}$$

$$\frac{dy}{dt} = y(4 - x - y) = 0 \Rightarrow \begin{cases} y = 0 & \text{or} \\ y = 4 - x & \end{cases}$$

The equilibria occur at the intersections of the nullclines, so in the case the equilibrium points are at $(0, 0)$, $(2, 0)$, $(0, 4)$, and $(3, 1)$.

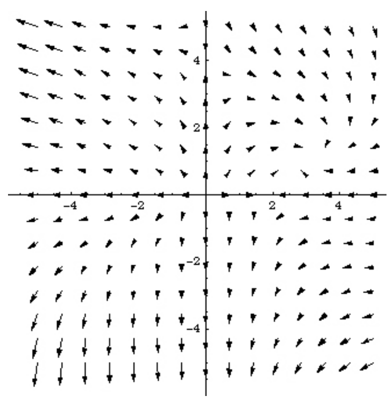


Figure 1: Phase portrait for Exercise 1a.

b. Let $f(x) = x(2 - x + y)$ and $g(x) = y(4 - x - y)$. Then

$$\mathbf{J} = \left\{ \begin{matrix} \frac{\partial f}{\partial x}(3,1) & \frac{\partial f}{\partial y}(3,1) \\ \frac{\partial g}{\partial y}(3,1) & \frac{\partial g}{\partial y}(3,1) \end{matrix} \right\} = \left\{ \begin{matrix} -3 & 3 \\ -1 & -1 \end{matrix} \right\}$$

c. We see that the eigenvalues of $\mathbf{J}$ at $(3, 1)$ are given by $\det(\mathbf{J}(3, 1) - \lambda I_2) = \lambda^2 + 4\lambda + 6 = 0 \Rightarrow \lambda = -2 \pm i\sqrt{2}$. Since both eigenvalues have negative real part, the equilibrium is stable.

2. a. Here we need to find the point where $\frac{dx}{dt} = \frac{dy}{dt} = 0$, where x and y are both nonzero. In this case we get

$$\frac{dx}{dt} = x(1 - x + ky - k) = 0 \Rightarrow 1 - x + ky - k = 0 \Rightarrow x = -k + ky + 1$$

$$\frac{dy}{dt} = y(1 - y + kx - k) = 0 \Rightarrow 1 - y + kx - k = 0$$

$$\Rightarrow 1 - y + k(-k + ky + 1) - k = 1 - k^2 + y(k^2 - 1) = 0$$

$$\Rightarrow y = 1, x = 1$$

So $(a, b) = (1, 1)$ is the equilibrium point we need.

b. The Jacobian at this point is

$$\mathbf{J} = \left\{\begin{matrix} \frac{\partial f}{\partial x}(1,1) & \frac{\partial f}{\partial y}(1,1) \\ \frac{\partial g}{\partial y}(1,1) & \frac{\partial g}{\partial y}(1,1) \end{matrix}\right\} = \left\{\begin{matrix} -1 & k \\ k & -1 \end{matrix}\right\}$$

c. The condition for stability is that both eigenvalues have negative real part. In this case we find that $\det(\mathbf{J}(1,1) - \lambda I_2) = \lambda^2 + 2\lambda + (1-k^2) = 0 \Rightarrow \lambda = -1 \pm \sqrt{k^2}$, so that in order for both eigenvalues to have negative real part we must have $|k| < 1$.

3. a. The nullclines are

$$\frac{d\alpha}{dt} = 0 \Rightarrow \omega = 0$$
$$\frac{d\omega}{dt} = 0 \Rightarrow \alpha = n\pi, \text{ for some } n \in \mathbf{Z}$$

The equilibrium points occur at $(\alpha, \omega) = (0, n\pi)$.

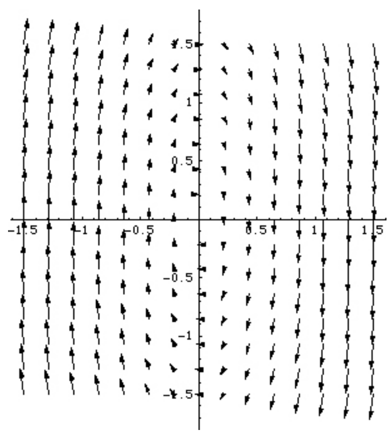


Figure 2: Phase portrait for Exercise 3a.

b. $\mathbf{J} = \left\{\begin{matrix} 0 & 1 \\ -\frac{g}{L} & 0 \end{matrix}\right\}$. The eigenvalues are $\lambda = \pm i\sqrt{\frac{g}{L}}$. This suggests that the system remains in closed orbits near the equilibrium points, so these points are not stable in the sense in which we have defined them to be.

4. Here the nullclines occur at

$$x^2 + y^2 - 1 = 0 \Rightarrow x^2 + y^2 = 1$$
$$xy = 0 \Rightarrow \begin{cases} x = 0, & \text{or} \\ y = 0 \end{cases}$$

The equilibria occur at the points $(\pm 1, 0), (0, \pm 1)$.

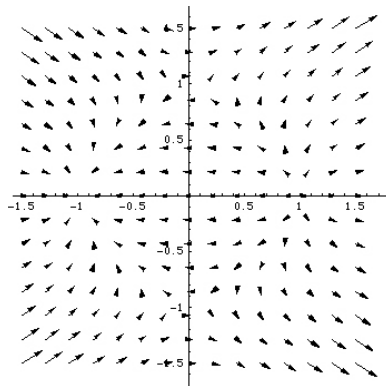


Figure 3: Phase portrait for Exercise 4.

We have that the Jacobian is

$$\mathbf{J} = \begin{Bmatrix} 2x & 2y \\ y & x \end{Bmatrix}$$

So that the eigenvalues are $\det(\mathbf{J} - \lambda I_2) = \lambda^2 - 3x\lambda + 2x^2 - 2y^2 = 0 \Rightarrow \lambda = \frac{1}{2}\left(3x \pm \sqrt{9x^2 - 4(2x^2 - 2y^2)}\right) = \frac{1}{2}\left(3x \pm \sqrt{x^2 + 8y^2}\right)$. The stable equilibiria occur when both eigenvalues have negative real part; we see that this occurs at $(-1, 0)$ only.

5. The nullclines of the system are

$$v = 0$$
$$f(u) - v = 0 \Rightarrow v = f(u) = \frac{du}{dt}$$

The only equilibrium point occurs when $(u, v) = (1, 0)$. Note that at this point $f(u) = 0$ and $f'(u) > 0$. Thus we see that

$$\det(\mathbf{J}(1,0) - \lambda I_2) = \begin{vmatrix} 0 - \lambda & 1 \\ f'(u = 1) & -1 - \lambda \end{vmatrix} = \lambda^2 + \lambda - f'(u = 1) = 0 \Rightarrow \lambda = \frac{1}{2}(-1 \pm \sqrt{1 + 4f'(1)})$$

Since $\sqrt{1 + 4f'(1)} > \sqrt{5} > 1$, one of the eigenvalues is positive and so this equilibrium is unstable.