Answers to exercises in Diff. Eq. Handout.

Section 10.1:

1. b) and c).

2. a), c).

3. a), c), d).

4. A basis is $\{e^{3t}, e^{-4t}\}$. $f(t) = \frac{1}{7}e^{3t} - \frac{1}{7}e^{-4t}$.

5. A basis is $\{e^{-t}\cos(t), e^{-t}\sin(t)\}$. $f(t) = e^{-t}\cos(t) + \frac{e-\cos(1)}{\sin(1)}e^{-t}\sin(t)$.

6. A basis is $\{e^{-3t}, t\,e^{-3t}\}$. $f(t) = -\frac{1}{4}(1-t)e^{-3t}$.

7. A basis is $\{t, -2+t^2\}$.

8. A basis is $\{1+t^2, t+t^2\}$.

9. If $f'(t) = \frac{1}{t}$, then general solution is $f(t) = c + \ln(t)$ which blows up as $t \to 0$ for any c.

10. a) $g'(t) = 2t\,f(t) + t^2\,f(t)$ and we are told that this is zero.
    b) Since $g'(t) = 0$, this function g is constant. Thus, $t^2 f(t) = c$ and $f(t) = c\frac{1}{t^2}$. This blows up at the origin if $c \neq 0$.
    c) Any function in the image must be zero at $t = 0$.

Section 10.2:

1. One can solve this problem using Graham-Schmid directly, or proceed as follows: The
first observation is that $\sqrt{\frac{1}{2}}$ has norm 1 and both $e^t - \frac{1}{\pi}\sinh(\pi)$ and $e^{-t} - \frac{1}{\pi}\sinh(\pi)$ are orthogonal to it. However, they are not orthogonal to each other. To remedy this, I note that the sum of the latter two,

$$\frac{1}{\alpha}(e^t - \frac{1}{\pi}\sinh(\pi)) + \frac{1}{\alpha}(e^{-t} - \frac{1}{\pi}\sinh(\pi)) = \frac{2}{\alpha}(\cosh(t) - \frac{1}{\pi}\sinh(\pi)),$$

and the difference,

$$\tfrac{1}{\alpha}(e^{t} - \tfrac{1}{\pi}\sinh(\pi)) - \tfrac{1}{\alpha}(e^{-t} - \tfrac{1}{\pi}\sinh(\pi)) = \tfrac{2}{\alpha}\sinh(t),$$

are orthogonal to each other as well as to $\sqrt{\tfrac{1}{2}}$. Thus, it is just a matter of dividing these last two functions by their norms. Doing so gives us the basis

$$\{\sqrt{\tfrac{1}{2}},\ a\,(\cosh(t) - \tfrac{1}{\pi}\sinh(\pi)),\ b\,\sinh(t)\}$$

where $a = (1+ \tfrac{1}{2\pi}\sinh(2\pi) - (\tfrac{1}{2\pi})^2 \sinh(2\pi))^{-1}$ and $b = (\tfrac{1}{2\pi}\sinh(2\pi) - 1)^{-1/2}$. The projection of t onto this basis is $c\,a^2 \sinh(t)$ where $c = 2\cosh(\pi) - \tfrac{2}{\pi}\sinh(\pi)$.

2. $|t| = \tfrac{\pi}{2} - \tfrac{4}{\pi}\sum_{k=1,3,5\ldots} \tfrac{1}{k^2}\cos(kt)$.

3. $\cosh(at) = \tfrac{1}{a\pi}(e^{a\pi} - e^{-a\pi})[\tfrac{1}{2} + \sum_{k=1}^{\infty} (-1)^k \tfrac{a^2}{a^2+k^2}\cos(kt)]$. Setting $t = \pi$ finds that

$$\sum_{k=1}^{\infty} \frac{1}{a^2+k^2} = \frac{1}{2a^2}(\pi a \coth(a\pi) - 1).$$

4. When evaluating the integrals, make the substitution $t \to t+r$. This changes each integral into the corresponding $r = 0$ integral.

5. When evaluating the integrals, make the substitution $t \to \tfrac{b-a}{2\pi}t + r$ to change each integral into the corresponding $r = 0$ integral.

Section 10.3:

1. $T(t, x) = \tfrac{1}{8} - e^{-4\mu t}\cos(2x) - \tfrac{1}{8}e^{-16\mu t}\cos(4x)$.

2. $T(t, x) = -\tfrac{1}{\pi}(e^{\pi} - e^{-\pi})\sum_{k=1}^{\infty} (-1)^k e^{-\mu k^2 t}\tfrac{k}{1+k^2}\sin(kx))]$.

3. The t-derivative is $\mu c^2 e^{\mu c^2 t} e^{cx}$ and the second x-derivative is $c^2 e^{\mu c^2 t} e^{cx}$.

4. The solution $e^{\mu t} e^{x}$ grows as $t \to \infty$, while the one in the text approaches 0 as $t \to \infty$.

5. The Fourier series solution is: $T(t, x) = 2\sum_{k=1}^{\infty} (-1)^{k+1} e^{-\mu k^2 t}\tfrac{1}{k}\sin(kx)$.

6. Since both the t-derivative of x and its second x-derivative are zero, it obeys the heat equation. It is not the one given above since one is time independent and the other is not.