Section 10.4:

1. a) $F(x,y) = \frac{1}{2}\sum_{k=1,2,\ldots} [\frac{b_k+a_k}{\cosh(k\pi)} \cosh(kx) + \frac{b_k-a_k}{\sinh(k\pi)} \sinh(kx)] \sin(ky)$.
   b) $G(x,y) = \frac{1}{2}\sum_{k=1,2,\ldots} [\frac{d_k+c_k}{\cosh(k\pi)} \cosh(ky) + \frac{d_k-c_k}{\sinh(k\pi)} \sinh(ky)] \sin(kx)$.
   c) $H(x, y) = F(x,y) + G(x,y)$.

2. a) The constant function $u(x,y) = c$.
   b) The temperature is 1.

3. a) The solution is $u(t,x) = \sum_{k=1,\ldots} \frac{1}{ck} b_k \sin(kct) \sin(kx)$.
   b) The solution is $u(t,x) = \sum_{k=1,\ldots} (a_k \cos(kct) + \frac{1}{ck} b_k \sin(kct)) \sin(kx)$.

4. a) Using the chain rule, $\frac{\partial f}{\partial t} = c \frac{\partial f}{\partial x}$ and $\frac{\partial^2 f}{\partial t^2} = c \frac{\partial}{\partial t}\frac{\partial f}{\partial x} = c^2 \frac{\partial^2 f}{\partial x^2}$. Meanwhile, $\frac{\partial g}{\partial t} = -c \frac{\partial g}{\partial x}$ and $\frac{\partial^2 g}{\partial t^2} = -c \frac{\partial}{\partial t}\frac{\partial g}{\partial x} = c^2 \frac{\partial^2 g}{\partial x^2}$.

   b) Set $y = -\pi - ct$. Then $ct = -\pi - y$ and so the condition $u(t,-\pi) = 0$ reads $f(-y-2\pi) + g(y) = 0$. Next, set $ct = \pi - y$ so that the condition $u(t,\pi) = 0$ reads as $f(2\pi-y) + g(y) = 0$. This last equation is one of the desired equations. To obtain the other, subtract the first from the second to find that $f(-2\pi-y) = - f(2\pi-y)$. Thus, setting $z = -2\pi-y$, this says $f(z) = -f(z+4\pi)$ which is the second of the desired equations. Having identified $g(y)$ as $f(-y-2\pi)$ the expression from $u(t,x)$ reads

$$u(t,x) = f(x+ct) + f(-(x-ct)-2\pi)$$

   and this is $f(x+ct) - f(2\pi-x+ct)$ if we use the fact that $f(y) = -f(y+4\pi)$.

   c) If you differentiate the expression in Part b) for u with respect to t and set $t = 0$, it read $c \frac{\partial f}{\partial x} + c \frac{\partial(f(2\pi - x))}{\partial x} = 0$. Thus, $c \frac{\partial}{\partial x}(f(x) + f(2\pi-x)) = 0$ and so $f(x) + f(2\pi-x)$ is a constant.

   d) $u(t,x) = \cos(\frac{1}{2}x) \cos(\frac{1}{2}ct) = \frac{1}{2}[\cos(\frac{1}{2}(x+ct)) + \cos(\frac{1}{2}(x-ct))]$