Chapter 1

1.1

6. $\left|\begin{array}{l} x+2y+3z=8 \\ x+3y+3z=10 \\ x+2y+4z=9 \end{array}\right| \xrightarrow[-I]{\;} \begin{array}{l} \\ -I \\ -I \end{array} \left|\begin{array}{r} x+2y+3z=8 \\ y=2 \\ z=1 \end{array}\right| \xrightarrow{-2(II)}$

$\left|\begin{array}{r} x+3z=4 \\ y=2 \\ z=1 \end{array}\right| \xrightarrow{-3(III)} \left|\begin{array}{l} x=1 \\ y=2 \\ z=1 \end{array}\right|$, so that $(x,y,z)=(1,2,1)$.

10. $\left|\begin{array}{r} x+2y+3z=1 \\ 2x+4y+7z=2 \\ 3x+7y+11z=8 \end{array}\right| \xrightarrow{\;} \begin{array}{l} \\ -2(I) \\ -3(I) \end{array} \left|\begin{array}{r} x+2y+3z=1 \\ z=0 \\ y+2z=5 \end{array}\right| \xrightarrow[(II)\leftrightarrow(III)]{} \left|\begin{array}{r} x+2y+3z=1 \\ y+2z=5 \\ z=0 \end{array}\right| \xrightarrow{-2(II)}$

$\left|\begin{array}{r} x-z=-9 \\ y+2z=5 \\ z=0 \end{array}\right| \xrightarrow[-2(III)]{+(III)} \left|\begin{array}{l} x=-9 \\ y=5 \\ z=0 \end{array}\right|$, so that $(x,y,z)=(-9,5,0)$.

0. The total demand for the product of Industry A is 1000 (the consumer demand) plus 0.1b (the demand from Industry B). The output a must meet this demand: $a=1000+0.1b$.

Setting up a similar equation for Industry B we obtain the system $\left|\begin{array}{l} a=1000+0.1b \\ b=780+0.2a \end{array}\right|$ or $\left|\begin{array}{r} a-0.1b=1000 \\ -0.2a+b=780 \end{array}\right|$, which yields the unique solution $a=1100$ and $b=1000$.

Let v be the speed of the boat relative to the water, and s be the speed of the stream; then the speed of the boat relative to the land is $v+s$ downstream and $v-s$ upstream.

Using the fact that (distance) = (speed)(time), we obtain the system $\left|\begin{array}{l} 8=(v+s)\frac{1}{3} \\ 8=(v-s)\frac{2}{3} \end{array}\right|$ ← downstream ← upstream

The solution is $v=18$ and $s=6$.

The system reduces to $\left|\begin{array}{r} x-3z=1 \\ y+2z=1 \\ (k^2-4)z=k-2 \end{array}\right|$

This system has a unique solution if $k^2-4\neq 0$, that is, if $k\neq\pm 2$.
If $k=2$ the last equation is $0=0$, and there will be infinitely many solutions.
If $k=-2$ the last equation is $0=-4$, and there will be no solutions.

b.

$x_4=0$

$x_3=2-2x_4=2$

$x_2=5-3x_3-7x_4=5-6=-1$

$x_1=-3-2x_2+x_3-4x_4=-3+2+2=1$,

so that $(x_1,x_2,x_3,x_4)=(1,-1,2,0)$

Let us start by reducing the system:

$\left|\begin{array}{r} x+2y+3z=39 \\ x+3y+2z=34 \\ 3x+2y+z=26 \end{array}\right| \xrightarrow{\;} \begin{array}{l} \\ -I \\ -3(I) \end{array} \left|\begin{array}{r} x+2y+3z=39 \\ y-z=-5 \\ -4y-8z=-91 \end{array}\right|$

Note that the last two equations are exactly those we get when we substitute $x=39-2y-3z$: either way, we end up with the system $\left|\begin{array}{r} y-z=-5 \\ -4y-8z=-91 \end{array}\right|$.

1.2.) 10. The system reduces to $\left|\begin{array}{ccccccc} x_1 & & & + & x_4 & = & 1 \\ & x_2 & & - & 3x_4 & = & 2 \\ & & x_3 & + & 2x_4 & = & -3 \end{array}\right| \longrightarrow \left|\begin{array}{l} x_1 = 1 - x_4 \\ x_2 = 2 + 3x_4 \\ x_3 = -3 - 2x_4 \end{array}\right|$

Let $x_4 = t$.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1-t \\ 2+3t \\ -3-2t \\ t \end{bmatrix},$$ where t is an arbitrary real number.

12. The system reduces to $\left|\begin{array}{ccccccccccc} x_1 & & & & + & 3.5x_5 & + & x_6 & = & 0 \\ & x_2 & & & + & x_5 & & & = & 0 \\ & & x_3 & & & & - & \frac{5}{3}x_6 & = & 0 \\ & & & x_4 & + & 3x_5 & + & x_6 & = & 0 \end{array}\right| \longrightarrow \left|\begin{array}{l} x_1 = -3.5x_5 - x_6 \\ x_2 = -x_5 \\ x_3 = \frac{5}{3}x_6 \\ x_4 = -3x_5 - x_6 \end{array}\right|.$

Let $x_5 = r$ and $x_6 = t$.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -3.5r - t \\ -r \\ \frac{5}{3}t \\ -3r - t \\ r \\ t \end{bmatrix}$$

29. Since the number of oxygen atoms remains constant, we must have $2a + b = 2c + 3d$.

Considering hydrogen and nitrogen as well, we obtain the system $\left|\begin{array}{ccccccc} 2a & + & b & = & 2c & + & 3d \\ & & 2b & = & c & + & d \\ a & & & = & c & + & d \end{array}\right|$ or

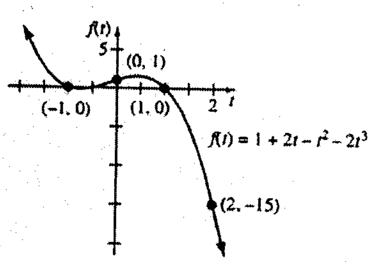
$\left|\begin{array}{ccccccccc} 2a & + & b & - & 2c & - & 3d & = & 0 \\ & & 2b & - & c & - & d & = & 0 \\ a & & & - & c & - & d & = & 0 \end{array}\right|$, which reduces to $\left|\begin{array}{cccccc} a & & & - & 2d & = 0 \\ & b & & - & d & = 0 \\ & & c & - & d & = 0 \end{array}\right|$.

The solutions are $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 2t \\ t \\ t \\ t \end{bmatrix}$.

To get the smallest positive integers, we set $t = 1$:
$2NO_2 + H_2O \longrightarrow HNO_2 + HNO_3$

30. Plugging the points into $f(t)$, we obtain the system $\left|\begin{array}{ccccccccc} a & & & & & & & = & 1 \\ a & + & b & + & c & + & d & = & 0 \\ a & - & b & + & c & - & d & = & 0 \\ a & + & 2b & + & 4c & + & 8d & = & -15 \end{array}\right|$

with unique solution $a = 1$, $b = 2$, $c = -1$, and $d = -2$, so that $f(t) = 1 + 2t - t^2 - 2t^3$.



32. The requirement $f_i'(a_i) = f_{i+1}'(a_i)$ and $f_i''(a_i) = f_{i+1}''(a_i)$ ensure that at each junction two different cubics fit "into" one another in a "smooth" way, since they must have the same slope and be equally curved. The requirement that $f_1'(a_0) = f_n'(a_n) = 0$ ensures that the track is horizontal at the beginning and at the end. How many unknowns are there? There are n pieces to be fit, and each one is a cubic of the form $f(t) = p + qt + rt^2 + st^3$, with p, q, r, and s to be determined; therefore, there are $4n$ unknowns. How many equations are there?

$f_i(a_i) = b_i$	for $i = 1, 2, \ldots, n$	gives n equations
$f_i(a_{i-1}) = b_{i-1}$	for $i = 1, 2, \ldots, n$	gives n equations
$f_i'(a_i) = f_{i+1}'(a_i)$	for $i = 1, 2, \ldots, n-1$	gives $n-1$ equations
$f_i''(a_i) = f_{i+1}''(a_i)$	for $i = 1, 2, \ldots, n-1$	gives $n-1$ equations
$f_1'(a_0) = 0, f_n'(a_n) = 0$		gives 2 equations

Altogether, we have $4n$ equations; convince yourself that all these equations are linear.

37. Compare with the solution of Exercise 1.1.21.

The diagram tells us that $\left| \begin{array}{l} x_1 = 0.2x_2 + 0.3x_3 + 320 \\ x_2 = 0.1x_1 + 0.4x_3 + 90 \\ x_3 = 0.2x_1 + 0.5x_2 + 150 \end{array} \right|$ or $\left| \begin{array}{r} x_1 - 0.2x_2 - 0.3x_3 = 320 \\ -0.1x_1 + x_2 - 0.4x_3 = 90 \\ -0.2x_1 - 0.5x_2 + x_3 = 150 \end{array} \right|$

This system has the unique solution $x_1 = 500$, $x_2 = 300$, and $x_3 = 400$.

40. We want to find m_1, m_2, m_3 such that $m_1 + m_2 + m_3 = 1$ and $\frac{1}{1}\left(m_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + m_2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + m_3 \begin{bmatrix} 4 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, that is, we have to solve the system $\left| \begin{array}{r} m_1 + m_2 + m_3 = 1 \\ m_1 + 2m_2 + 4m_3 = 2 \\ 2m_1 + 3m_2 + m_3 = 2 \end{array} \right|$.

The unique solution is $m_1 = \frac{1}{2}$, $m_2 = \frac{1}{4}$, and $m_3 = \frac{1}{4}$.

We will put $\frac{1}{2}$ kg at the point $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\frac{1}{4}$ kg at each of the two other vertices.

1, 3) 4. This matrix has rank 2 since its rref is $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

14. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = -1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + 1 \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$ or $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} =$
$\begin{bmatrix} 1 \cdot (-1) + 2 \cdot 2 + 3 \cdot 1 \\ 2 \cdot (-1) + 3 \cdot 2 + 4 \cdot 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$

24. By Fact 1.3.4, rref $(A) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Using Fact 1.3.4 again, we can conclude that the system $A\vec{x} = \vec{c}$ has a unique solution as well.

25. In this case, rref(A) has a row of zeros, so that rank$(A) < 4$; there will be a nonleading variable. The system $A\vec{x} = \vec{c}$ could have infinitely many solutions (for example, when $\vec{c} = \vec{0}$) or no solutions (for example, when $\vec{c} = \vec{b}$), but it cannot have a unique solution, by Fact 1.3.4.

36. By Exercise 35, the ith column of A is $A\vec{e}_i$, for $i = 1, 2, 3$. Therefore, $A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$.

50. The right-most column of rref$[A \vdots \vec{b}]$ must contain a leading one, so that the system has no solutions.

57. Pick a vector on each line, say $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ on $y = \frac{x}{2}$ and $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ on $y = 3x$.

Then write $\begin{bmatrix} 7 \\ 11 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$: $a \begin{bmatrix} 2 \\ 1 \end{bmatrix} + b \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ 11 \end{bmatrix}$.

The unique solution is $a = 2$, $b = 3$, so that the desired representation is $\begin{bmatrix} 7 \\ 11 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 9 \end{bmatrix}$.

$\begin{bmatrix} 4 \\ 2 \end{bmatrix}$ is on the line $y = \frac{x}{2}$; $\begin{bmatrix} 3 \\ 9 \end{bmatrix}$ is on line $y = 3x$.