2.1.

6. Note that $x_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, so that T is indeed linear, with matrix $\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$.

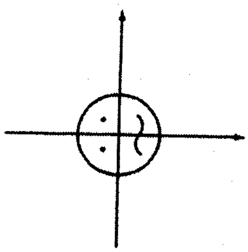
14. a. By Exercise 13(a) $\begin{bmatrix} 2 & 3 \\ 5 & k \end{bmatrix}$ is invertible if (and only if) $2k - 15 \neq 0$, or $k \neq 7.5$.

b. By Exercise 13(b), $\begin{bmatrix} 2 & 3 \\ 5 & k \end{bmatrix}^{-1} = \frac{1}{2k-15} \begin{bmatrix} k & -3 \\ -5 & 2 \end{bmatrix}$.

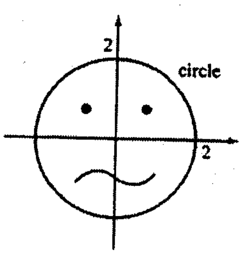
If all entries of this inverse are integers, then $\frac{3}{2k-15} - \frac{2}{2k-15} = \frac{1}{2k-15}$ is a (nonzero) integer n, so that $2k - 15 = \frac{1}{n}$ or $k = 7.5 + \frac{1}{2n}$. Since $\frac{k}{2k-15} = kn = 7.5n + \frac{1}{2}$ is an integer as well, n must be odd.

We have shown: If all entries of the inverse are integers, then $k = 7.5 + \frac{1}{2n}$, where n is an odd integer. The converse is true as well: If k is chosen in this way, then the entries of $\begin{bmatrix} 2 & 3 \\ 5 & k \end{bmatrix}^{-1}$ will be

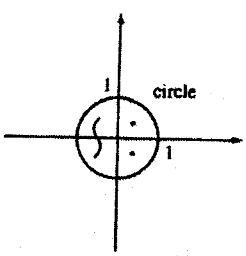
24. Compare with Example 5.



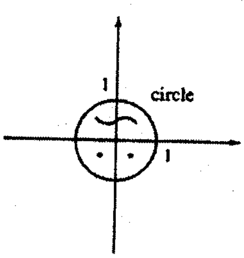
25. The matrix represents a dilation by the factor of 2.



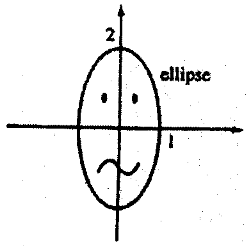
26. Matrix represents a reflection in the line $x_2 = x_1$.



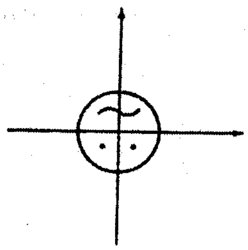
27. Matrix represents a reflection in the $\vec{e}_1$ axis.



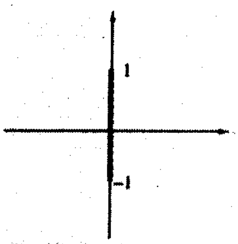
28. If $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, then $A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 2x_2 \end{bmatrix}$, so that the x_2 component is multiplied by 2, while the x_1 component remains unchanged.



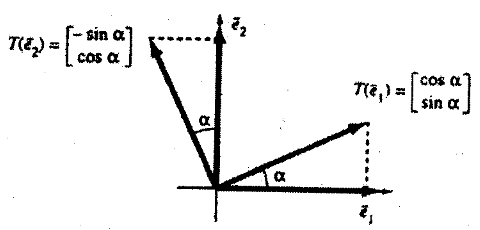
29. Matrix represents a reflection in the origin. Compare with Exercise 17.



30. If $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, then $A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \end{bmatrix}$, so that A represents the projection onto the $\vec{e}_2$ axis.

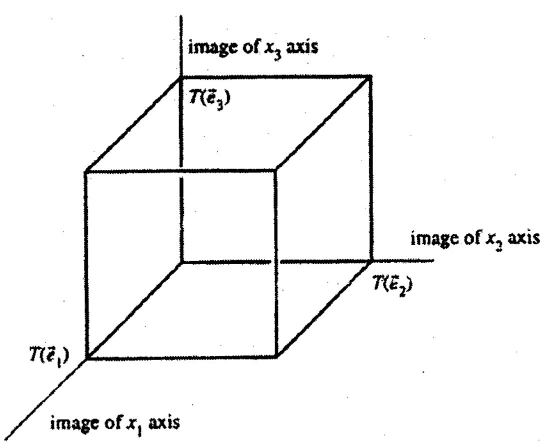


34. As in Exercise 33, we find $T(\vec{e}_1)$ and $T(\vec{e}_2)$; then by Fact 2.1.2, $A = [T(\vec{e}_1)\ T(\vec{e}_2)]$.



Therefore, $A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$.

42. a.



b. Solve the equation $\begin{bmatrix} -\frac{1}{2} & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, or $\left| \begin{array}{rrrl} -\frac{1}{2}x_1 & + & x_2 & = 0 \\ -\frac{1}{2}x_1 & + & x_3 & = 0 \end{array} \right|$.

$\begin{bmatrix} x_1 \end{bmatrix}$ $\begin{bmatrix} 2t \end{bmatrix}$

b. Solve the equation $\begin{bmatrix} -\frac{1}{2} & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, or $\left| \begin{array}{lllll} -\frac{1}{2}x_1 & + & x_2 & & = 0 \\ -\frac{1}{2}x_1 & + & & x_3 & = 0 \end{array} \right|$.

The solutions are of the form $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2t \\ t \\ t \end{bmatrix}$, where t is an arbitrary real number. These points are on the line through the origin and the observer's eye.

43. a. $T(\vec{x}) = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 2x_1 + 3x_2 + 4x_3 = [2\ 3\ 4] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

The transformation is indeed linear, with matrix $[2\ 3\ 4]$.

b. If $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$, then T is linear with matrix $[v_1\ v_2\ v_3]$, as in part (a).

c. Let $[a\ b\ c]$ be the matrix of T. Then $T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = [a\ b\ c] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = ax_1 + bx_2 + cx_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, so that $\vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ does the job.

44. $T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} v_2x_3 - v_3x_2 \\ v_3x_1 - v_1x_3 \\ v_1x_2 - v_2x_1 \end{bmatrix} = \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, so that T is linear, with matrix $\begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix}$.

2.2

6. By Fact 2.2.5, $\text{proj}_L \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \left(\vec{u} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) \vec{u}$, where $\vec{u}$ is a unit vector on L. To get $\vec{u}$, we normalize $\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$:

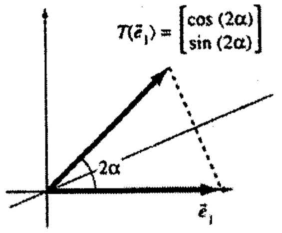
$\vec{u} = \frac{1}{3} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$, so that $\text{proj}_L \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{5}{3} \cdot \frac{1}{3} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{10}{9} \\ \frac{5}{9} \\ \frac{10}{9} \end{bmatrix}$.

15. Proceeding as in Exercise 13, we find that A is the matrix whose ijth entry is $2u_iu_j$ if $i \neq j$ and $u_i^2 - 1$ if $i = j$:

$$A = \begin{bmatrix} u_1^2 - 1 & 2u_1u_2 & \dots & 2u_1u_n \\ 2u_2u_1 & u_2^2 - 1 & \dots & 2u_2u_n \\ \vdots & \vdots & \ddots & \vdots \\ 2u_nu_1 & 2u_nu_2 & \dots & u_n^2 - 1 \end{bmatrix}$$

16. a.

16 b. By Fact 2.1.2, the matrix of T is $[T(\vec{e}_1) \quad T(\vec{e}_2)]$.



$T(\vec{e}_2)$ is the unit vector in the fourth quadrant perpendicular to $T(\vec{e}_1) = \begin{bmatrix} \cos(2\alpha) \\ \sin(2\alpha) \end{bmatrix}$, so that $T(\vec{e}_2) = \begin{bmatrix} \sin(2\alpha) \\ -\cos(2\alpha) \end{bmatrix}$. The matrix of T is therefore $\begin{bmatrix} \cos(2\alpha) & \sin(2\alpha) \\ \sin(2\alpha) & -\cos(2\alpha) \end{bmatrix}$.

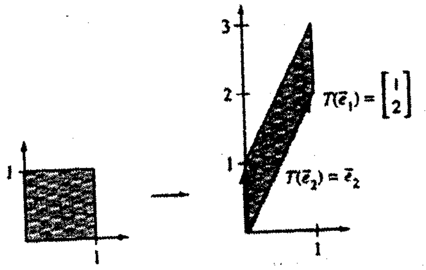
Alternatively, we can use the result of Exercise 13, with $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix}$ to find the matrix $\begin{bmatrix} 2\cos^2\alpha - 1 & 2\cos\alpha\sin\alpha \\ 2\cos\alpha\sin\alpha & 2\sin^2\alpha - 1 \end{bmatrix}$.

You can use trigonometric identities to show that the two results agree.

18. If T is indeed a reflection-dilation, then the factor of dilation is $\|T(\vec{e}_1)\| = \left\| \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\| = 5$, since a reflection leaves the length of a vector unchanged.

Now write $\begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix} = 5 \begin{bmatrix} 0.6 & 0.8 \\ 0.8 & -0.6 \end{bmatrix}$; we need to verify that the matrix $\begin{bmatrix} 0.6 & 0.8 \\ 0.8 & -0.6 \end{bmatrix}$ represents a reflection. The result of Exercise 13 shows that this is indeed the case, with $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}$.

24. a. Compare the Example 6.



b. We claim that T is a shear parallel to the $\vec{e}_2$ axis L. We need to check the two parts of Definition 2.2.4:

$A\vec{v} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ v_2 \end{bmatrix} = \vec{v}$, for all $\vec{v}$ in L, and $A\vec{x} - \vec{x} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 2x_1 + x_2 \end{bmatrix} - \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2x_2 \end{bmatrix}$ is on L for all $\vec{x}$ in $\mathbb{R}^2$.

c. The matrix of the inverse transformation is $\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$, representing a shear parallel to the $\vec{e}_2$ axis as well.

34. Keep in mind that the columns of the matrix of a linear transformation T are $T(\vec{e}_1)$, $T(\vec{e}_2)$, and $T(\vec{e}_3)$. If T is the orthogonal projection onto a line L, then $T(\vec{x})$ will be on L for all $\vec{x}$ in $\mathbb{R}^3$; in particular, the three columns of the matrix of T will be on L, and therefore pairwise parallel. This is the case only for matrix B: B represents an orthogonal projection onto a line.
A reflection transforms orthogonal vectors into orthogonal vectors; therefore, the three columns of its matrix must be pairwise orthogonal. This is the case only for matrix E: E represents the reflection in a line.

45. Since $\vec{v}_1$ and $\vec{v}_2$ are not parallel, any vector $\vec{x}$ in $\mathbb{R}^2$ can be written as a linear combination of $\vec{v}_1$ and $\vec{v}_2$: $\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2$.
Then $T(\vec{x}) = T(c_1\vec{v}_1 + c_2\vec{v}_2) = c_1T(\vec{v}_1) + c_2T(\vec{v}_2) = c_1L(\vec{v}_1) + c_2L(\vec{v}_2) = L(c_1\vec{v}_1 + c_2\vec{v}_2) = L(\vec{x})$, as claimed.

unit vector perpendicular to L, and set