12. Use Fact 2.3.5; the inverse is $\begin{bmatrix} 5 & -20 & -2 & -7 \\ 0 & -1 & 0 & 0 \\ -2 & 6 & 1 & 2 \\ 0 & 3 & 0 & 1 \end{bmatrix}$

20. Solving for x_1, x_2, and x_3 in terms of y_1, y_2, and y_3 we find that

$$x_1 = -8y_1 - 15y_2 + 12y_3$$
$$x_2 = 4y_1 + 6y_2 - 5y_3$$
$$x_3 = -y_1 - y_2 + y_3$$

30. Use Fact 2.3.3:

$$\begin{bmatrix} 0 & 1 & b \\ -1 & 0 & c \\ -b & -c & 0 \end{bmatrix} \xrightarrow{I \leftrightarrow II} \begin{bmatrix} -1 & 0 & c \\ 0 & 1 & b \\ -b & -c & 0 \end{bmatrix} \xrightarrow{\div(-1)} \begin{bmatrix} 1 & 0 & -c \\ 0 & 1 & b \\ -b & -c & 0 \end{bmatrix} \begin{matrix} \\ \\ +b(I)+c(II) \end{matrix} \longrightarrow \begin{bmatrix} 1 & 0 & -c \\ 0 & 1 & b \\ 0 & 0 & 0 \end{bmatrix}$$

This matrix is noninvertible, regarless of the values of b and c.

38. Use Fact 2.3.6; $A^{-1} = \frac{1}{-1}\begin{bmatrix} -1 & -k \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & k \\ 0 & -1 \end{bmatrix} (= A)$.

40. If you apply an elementary row operation to a matrix with two equal columns, then the resulting matrix will also have two equal columns. Therefore, rref(A) has two equal columns, so that rref(A) $\neq I_n$. Now use Fact 2.3.3.

41. a. Invertible: the transformation is its own inverse.

b. Not invertible: the equation $T(\vec{x}) = \vec{b}$ has infinitely many solutions if $\vec{b}$ is on the plane, and none otherwise.

c. Invertible: The inverse is dilation by $\frac{1}{5}$ (that is, a contraction by 5). If $\vec{y} = 5\vec{x}$, then $\vec{x} = \frac{1}{5}\vec{y}$.

d. Invertible: The inverse is a rotation about the same axis through the same angle in the opposite direction.

44. a. $\text{rref}(M_4) = \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, so that rank$(M_4) = 2$.

b. To simplify the notation, we introduce the row vectors $\vec{v} = [1\ 1\ \ldots\ 1]$ and $\vec{w} = [0\ n\ 2n\ \ldots\ (n-1)n]$ with n components. Then we can write M_n in terms of its rows as $M_n = \begin{bmatrix} \vec{v}+\vec{w} \\ 2\vec{v}+\vec{w} \\ \cdots \\ n\vec{v}+\vec{w} \end{bmatrix} \begin{matrix} \\ -2(I) \\ \cdots \\ -n(I) \end{matrix}$.

Applying the Gauss-Jordan algorithm to the first column we get $\begin{bmatrix} \vec{v}+\vec{w} \\ -\vec{w} \\ -2\vec{w} \\ \cdots \\ -(n-1)\vec{w} \end{bmatrix}$.

All the rows below the second are scalar multiples of the second; therefore, rank$(M_n) = 2$.

c. By part (b), the matrix M_n is invertible only if $n = 1$ or $n = 2$.

$A^2 = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$, $BC = [14 \ \ 8 \ \ 2]$, $BD = [6]$, $C^2 = \begin{bmatrix} -2 & -2 & -2 \\ 4 & 1 & -2 \\ 10 & 4 & -2 \end{bmatrix}$, $CD = \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix}$, $DB = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$,

$DE = \begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix}$, $EB = [5 \ 10 \ 15]$, $E^2 = [25]$

28. $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is one such matrix.

33. By Fact 1.3.3, there is a nonzero $\vec{x}$ such that $B\vec{x} = \vec{0}$ and therefore $AB\vec{x} = \vec{0}$. By Fact 2.3.4b, the 3×3 matrix AB is not invertible.

38. We need to find all matrices $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, or

$$\begin{bmatrix} a & 2a+3b \\ c & 2c+3d \end{bmatrix} = \begin{bmatrix} a+2c & b+2d \\ 3c & 3d \end{bmatrix}.$$

The condition $c = 3c$ implies that $c = 0$. There is one other constraint: $2a + 3b = b + 2d$, or $d = a + b$.

This means that X must be of the form $X = \begin{bmatrix} a & b \\ 0 & a+b \end{bmatrix}$, where a and b are arbitrary.

56. Performing a sequence of p elementary row operations on a matrix A amounts to multiplying A with $E_1E_2 \dots E_p$ from the left, where the E_i are elementary matrices. If $I_n = E_1E_2 \dots E_pA$, then $E_1E_2 \dots E_p = A^{-1}$, so that

a. $E_1E_2 \dots E_pAB = B$, and

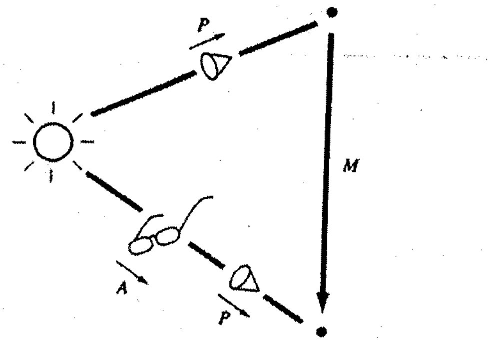
b. $E_1E_2 \dots E_pI_n = A^{-1}$.

76. a. $\begin{aligned} I &= \tfrac{1}{3}R + \tfrac{1}{3}G + \tfrac{1}{3}B \\ L &= R - G \\ S &= -\tfrac{1}{2}R - \tfrac{1}{2}G + B \end{aligned}$, so that the matrix is $P = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 1 & -1 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix}$.

b. $\begin{bmatrix} R \\ G \\ B \end{bmatrix}$ is transformed into $\begin{bmatrix} R \\ G \\ 0 \end{bmatrix}$, with matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

c. This matrix is $PA = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & 0 \\ 1 & -1 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix}$ (we apply first A, then P.)

d.



A "diagram chase" shows that $M = PAP^{-1} = \begin{bmatrix} \frac{2}{3} & 0 & -\frac{2}{9} \\ 0 & 1 & 0 \\ -1 & 0 & \frac{1}{3} \end{bmatrix}$.

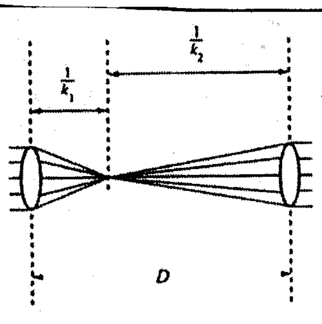
80. a. The formula $\begin{bmatrix} y \\ n \end{bmatrix} = \begin{bmatrix} 1 - Rk & L + R - kLR \\ -k & 1 - kL \end{bmatrix} \begin{bmatrix} x \\ m \end{bmatrix}$ is given, which implies that $y = (1 - Rk)x + (L + R - kLR)m$.
In order for y to be independent of x it is required that $1 - Rk = 0$, or $k = \frac{1}{R} = 40$ (diopters). $\frac{1}{k}$ then equals R, which is the distance between the plane of the lens and the plane on which parallel incoming rays focus at a point; thus the term "focal length" for $\frac{1}{k}$.

b. Now we want y to be independent of the slope m (it must depend on x alone). In view of the formula above, this is the case if $L + R - kLR = 0$, or $k = \dfrac{L+R}{LR} = \dfrac{1}{R} + \dfrac{1}{L} = 40 + \dfrac{10}{3} \approx 43.3$ (diopters).

c. Here the transformation is

$$\begin{bmatrix} y \\ n \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -k_1 & 1 \end{bmatrix} \begin{bmatrix} 1 & D \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -k_1 & 1 \end{bmatrix} \begin{bmatrix} x \\ m \end{bmatrix} = \begin{bmatrix} 1 - k_1 D & D \\ k_1 k_2 D - k_1 - k_2 & 1 - k_2 D \end{bmatrix} \begin{bmatrix} x \\ m \end{bmatrix}.$$

We want the slope n of the outgoing rays to depend on the slope m of the incoming rays alone, and not on x; this forces $k_1 k_2 D - k_1 - k_2 = 0$, or, $D = \dfrac{k_1 + k_2}{k_1 k_2} = \dfrac{1}{k_1} + \dfrac{1}{k_2}$, the sum of the focal lengths of the two lenses.



2. Find all $\vec{x}$ such that $A\vec{x} = \vec{0}$:

$$\left[\begin{array}{cc:c} 2 & 3 & 0 \\ 6 & 9 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc:c} 1 & \frac{3}{2} & 0 \\ 0 & 0 & 0 \end{array}\right], \text{ so that } \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{3t}{2} \\ t \end{bmatrix}$$

Setting $t = 2$ we find $\ker(A) = \operatorname{span}\begin{bmatrix} -3 \\ 2 \end{bmatrix}$.

20. Since the three column vectors of A are parallel, we have $\operatorname{im}(A) = \operatorname{span}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, a line in $\mathbb{R}^3$.

32. By Fact 3.1.3, $A = \begin{bmatrix} 7 \\ 6 \\ 5 \end{bmatrix}$ does the job. There are many other correct answers: any nonzero $3 \times n$ matrix A whose column vectors are scalar multiples of $\begin{bmatrix} 7 \\ 6 \\ 5 \end{bmatrix}$.

33. The plane is the kernel of the linear transformation $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = x + 2y + 3z$ from $\mathbb{R}^3$ to $\mathbb{R}$.

34. To describe a subset of $\mathbb{R}^3$ as a kernel means to describe it as an intersection of planes (think about it). By inspection, the given line is the intersection of the planes

$x + y = 0$ and
$2x + z = 0$.

This means that the line is the kernel of the linear transformation $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + y \\ 2x + z \end{bmatrix}$ from $\mathbb{R}^3$ to $\mathbb{R}^2$.

37. $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, $A^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $A^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, so that

$\ker(A) = \operatorname{span}(\vec{e}_1)$, $\ker(A^2) = \operatorname{span}(\vec{e}_1, \vec{e}_2)$, $\ker(A^3) = \mathbb{R}^3$, and
$\operatorname{im}(A) = \operatorname{span}(\vec{e}_1, \vec{e}_2)$, $\operatorname{im}(A^2) = \operatorname{span}(\vec{e}_1)$, $\operatorname{im}(A^3) = \{\vec{0}\}$.

44. a. Yes; by construction of the echelon form, the systems $A\vec{x} = \vec{0}$ and $B\vec{x} = \vec{0}$ have the same solutions (it is the whole point of Gaussian elimination not to change the solutions of a system).

b. No; as a counterexample, consider $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, with $\operatorname{im}(A) = \operatorname{span}(\vec{e}_2)$, but $B = \operatorname{rref}(A) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, with $\operatorname{im}(B) = \operatorname{span}(\vec{e}_1)$.

50. From Exercise 38 we know that $\ker(A^3) \subseteq \ker(A^4)$. Conversely, if $\vec{x}$ is in $\ker(A^4)$, then $A^4\vec{x} = A^3(A\vec{x}) = \vec{0}$, so that $A\vec{x}$ is in $\ker(A^3) = \ker(A^2)$, which implies that $A^2(A\vec{x}) = A^3\vec{x} = \vec{0}$, that is, $\vec{x}$ is in $\ker(A^3)$. We have shown that $\ker(A^3) = \ker(A^4)$.

54. a. If no error occurred, then $\vec{w} = \vec{v} = M\vec{u}$, and $H\vec{w} = H(M\vec{u}) = \vec{0}$, by Exercise 53b.
If an error occurred in the ith component, then $\vec{w} = \vec{v} + \vec{e}_i = M\vec{u} + \vec{e}_i$, so that
$H\vec{w} = H(M\vec{u}) + H\vec{e}_i = i$th column of H.
Since the columns of H are all different, this method allows us to find out where an error occurred.

b. $H\vec{w} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ = seventh column of H: an error occurred in the seventh component of $\vec{v}$.

Therefore $\vec{v} = \vec{w} + \vec{e}_7 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ and $\vec{u} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$.