18. Linearly dependent, since rref $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 7 \\ 1 & 4 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

24. We need to find all vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ in $\mathbb{R}^3$ such that $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = x + 2y + 3z = 0$.

These vectors have the form $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2s - 3t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$.

Therefore, $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$ is a basis of $L^{\perp}$.

32. The two column vectors $\vec{v}_1$ and $\vec{v}_2$ of the given matrix A are parallel, with $\vec{v}_2 = 2\vec{v}_1$, so that $\text{im}(A) = \text{span}(\vec{v}_1, \vec{v}_2) = \text{span}(\vec{v}_1)$.

Therefore, $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ is a basis of $\text{im}(A)$.

38. a. Consider vectors $\vec{v}_1, \ldots, \vec{v}_m$ in $\mathbb{R}^n$, where $m > n$. The rank of the $n \times m$ matrix $[\vec{v}_1 \ \vec{v}_2 \ \ldots \ \vec{v}_m]$ is at most n, so that there is at least one column without a leading one. By Fact 3.2.6, the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ will be linearly dependent.

b. Using the terminology introduced in the exercise, we need to show that any vector $\vec{v}$ in V is a linear combination of $\vec{v}_1, \ldots, \vec{v}_m$. Choose a specific vector $\vec{v}$ in V. Since we can find no more than m linearly independent vectors in V, the $m + 1$ vectors $\vec{v}_1, \ldots, \vec{v}_m, \vec{v}$ will be linearly dependent, that is, there will be a nontrivial relation $c_1 \vec{v}_1 + \cdots + c_m \vec{v}_m + c\vec{v} = \vec{0}$.
We must have $c \neq 0$ (if c were zero, we could drop the term $c\vec{v}$ in the relation above to get a nontrivial relation among $\vec{v}_1, \ldots, \vec{v}_m$, contradicting the linear independence of these vectors.) Now $\vec{v} = -\frac{c_1}{c} \vec{v}_1 - \cdots - \frac{c_m}{c} \vec{v}_m$. We were able to express $\vec{v}$ as a linear combination of $\vec{v}_1, \ldots, \vec{v}_m$, as claimed.

42. We can use the hint and form the dot product of $\vec{v}_i$ and both sides of the relation $c_1 \vec{v}_1 + \cdots + c_i \vec{v}_i + \cdots + c_m \vec{v}_m = \vec{0}$:
$(c_1 \vec{v}_1 + \cdots + c_i \vec{v}_i + \cdots + c_m \vec{v}_m) \cdot \vec{v}_i = \vec{0} \cdot \vec{v}_i$, so that $c_1 (\vec{v}_1 \cdot \vec{v}_i) + \cdots + c_i (\vec{v}_i \cdot \vec{v}_i) + \cdots + c_m (\vec{v}_m \cdot \vec{v}_i) = 0$.
Since $\vec{v}_i$ is perpendicular to all the other $\vec{v}_j$, we will have $\vec{v}_i \cdot \vec{v}_j = 0$ whenever $j \neq i$; since $\vec{v}_i$ is a unit vector, we will have $\vec{v}_i \cdot \vec{v}_i = 1$. Therefore, the equation above simplifies to $c_i = 0$.
Since this reasoning applies to all $i = 1, \ldots, m$, we have only the trivial relation among the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$, so that these vectors are linearly independent, as claimed.

46. Solve the system $\begin{vmatrix} x_1 & + & 2x_2 & & + & 3x_4 & + & 5x_5 & = 0 \\ & & & x_3 & + & 4x_4 & + & 6x_5 & = 0 \end{vmatrix}$.

The solutions are of the form $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_1 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2s - 3t - 5r \\ s \\ -4t - 6r \\ t \\ r \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ -4 \\ 1 \\ 0 \end{bmatrix} + r \begin{bmatrix} -5 \\ 0 \\ -6 \\ 0 \\ 1 \end{bmatrix}$.

The three vectors $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ -6 \\ 0 \\ 1 \end{bmatrix}$ span the kernel, by construction, and none of them is a linear combination of the two others (since each has a one as a component where the other two have a zero; consider the second, fourth, and fifth components). Therefore, the three vectors form a basis of the kernel.

50. The verification of the three properties listed in Definition 3.2.1 is straightforward. Alternatively, we can choose a basis $\vec{v}_1, \ldots, \vec{v}_p$ of V and a basis $\vec{w}_1, \ldots, \vec{w}_q$ of W (see Exercise 38b) and show that $V + W = \text{span}(\vec{v}_1, \ldots, \vec{v}_p, \vec{w}_1, \ldots, \vec{w}_q)$ (compare with Exercise 4).

Indeed, if $\vec{v} + \vec{w}$ is in $V + W$, then $\vec{v}$ is a linear combination of $\vec{v}_1, \ldots, \vec{v}_p$ and $\vec{w}$ is a linear combination of $\vec{w}_1, \ldots, \vec{w}_q$, so that $\vec{v} + \vec{w}$ is a linear combination of $\vec{v}_1, \ldots, \vec{v}_p, \vec{w}_1, \ldots, \vec{w}_q$. Conversely, if $\vec{x}$ is in $\text{span}(\vec{v}_1, \ldots, \vec{v}_p, \vec{w}_1, \ldots, \vec{w}_q)$, then $\vec{x} = (c_1 \vec{v}_1 + \cdots + c_p \vec{v}_p) + (d_1 \vec{w}_1 + \cdots + d_q \vec{w}_q)$, so that $\vec{x}$ is in $V + W$. If V and W are distinct lines in $\mathbb{R}^3$ (spanned by $\vec{v}$ and $\vec{w}$, respectively), then $V + W$ is the plane spanned by $\vec{v}$ and $\vec{w}$.

51. a. Consider a relation $c_1 \vec{v}_1 + \cdots + c_p \vec{v}_p + d_1 \vec{w}_1 + \cdots + d_q \vec{w}_q = \vec{0}$.

22. $\text{rref}(A) = \begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

A basis of $\text{im}(A)$ is $\begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ -1 \end{bmatrix}$, so that $\dim(\text{im}(A)) = 3$.

A basis of $\text{ker}(A)$ is $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$, so that $\dim(\text{ker}(A)) = 2$.

$\dim(\text{im}(A)) + \dim(\text{ker}(A)) = 5 = \#$ of columns, in accordance with Fact 3.3.9.

30. Proceeding as in Exercise 29, we find the basis $\begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

32. We need to find all vectors $\vec{x}$ in $\mathbb{R}^4$ such that $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} = 0$ and $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} = 0$.

This amounts to solving the system $\begin{vmatrix} x_1 & - & x_3 & + & x_4 & = & 0 \\ & x_2 & + & 2x_3 & + & 3x_4 & = & 0 \end{vmatrix}$, which in turn amounts to

finding the kernel of $\begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix}$.

Using the standard approach, we find the basis $\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 0 \\ 1 \end{bmatrix}$.

38. a. The rank of a 3×5 matrix A is 0,1,2, or 3, so that $\dim(\text{ker}(A)) = 5 - \text{rank}(A)$ is 2,3,4, or 5.

b. The rank of a 7×4 matrix A is at most 4, so that $\dim(\text{im}(A)) = \text{rank}(A)$ is 0,1,2,3, or 4.

44. Suppose that $V \cap W = \{\vec{0}\}$ and $\dim(V) + \dim(W) = n$.

Choose a basis $\vec{v}_1, \ldots, \vec{v}_p$ of V and a basis $\vec{w}_1, \ldots, \vec{w}_q$ in W; note that $p + q = n$. By Exercise 3.2.51b, the n vectors $\vec{v}_1, \ldots, \vec{v}_p, \vec{w}_1, \ldots, \vec{w}_q$ in $\mathbb{R}^n$ are linearly independent, so that they form a basis of $\mathbb{R}^n$ (by parts (vii) and (ix) of Summary 3.3.11). By Fact 3.2.7, any vector $\vec{x}$ can be written uniquely as

$\vec{x} = (c_1\vec{v}_1 + \cdots + c_p\vec{v}_p) + (d_1\vec{w}_1 + \cdots + d_q\vec{w}_q)$, with $\vec{v} = c_1\vec{v}_1 + \cdots + c_p\vec{v}_p$ in V and $\vec{w} = d_1\vec{w}_1 + \cdots + d_q\vec{w}_q$ in W, which gives the desired representation.

Conversely, suppose V and W are complements. Let us first show that $V \cap W = \{\vec{0}\}$ in this case. Indeed, if $\vec{x}$ is in $V \cap W$, then we can write $\vec{x} = \vec{x} + \vec{0} = \vec{0} + \vec{x}$
$$\uparrow \quad \uparrow \quad \uparrow \quad \uparrow$$
$$\text{in} \quad \text{in} \quad \text{in} \quad \text{in}$$
$$V \quad W \quad V \quad W$$

Since this representation is unique (by definition of complements), we must have $\vec{x} = \vec{0}$, so that $V \cap W = \{\vec{0}\}$. By definition of complements, we have $\mathbb{R}^n = V + W$, so that $n = \dim(V + W) = \dim(V) + \dim(W)$, by Exercise 43.

53. Using the terminology suggested in the hint, we observe that the vectors $\vec{v}, A\vec{v}, \ldots, A^n\vec{v}$ are linearly dependent (by Fact 3.3.4a), so that there is a nontrivial relation $c_0\vec{v} + c_1 A\vec{v} + \cdots + c_n A^n\vec{v} = \vec{0}$.
We can rewrite this relation in the form $(c_0 I_n + c_1 A + \cdots + c_n A^n)\vec{v} = \vec{0}$.
The nonzero vector $\vec{v}$ is in the kernel of the matrix $c_0 I_n + c_1 A + \cdots + c_n A^n$, so that this matrix is not invertible.

56. Using the terminology suggested in the Exercise, we multiply the relation $c_0\vec{v} + c_1 A\vec{v} + \cdots + c_{m-1}A^{m-1}\vec{v} = \vec{0}$ with A^{m-1} and obtain $c_0 A^{m-1}\vec{v} = \vec{0}$ (all other terms vanish since $A^m = 0$).
Since the vector $A^{m-1}\vec{v}$ is nonzero (by construction), the scalar c_0 must be zero, and our relation simplifies to $c_1 A\vec{v} + c_2 A^2\vec{v} + \cdots + c_{m-1}A^{m-1}\vec{v} = \vec{0}$.
Now we multiply both sides with A^{m-2} and obtain $c_1 A^{m-1}\vec{v} = \vec{0}$, so that $c_1 = 0$ as above. Continuing like this we conclude that all the c_i must be zero, as claimed.

4. Proceeding as in Example 1, we find $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$.

14. $A = \begin{bmatrix} 7 & -1 \\ -6 & 8 \end{bmatrix}$, $S = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$, $S^{-1} = \frac{1}{5}\begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}$.

By Fact 3.4.4 the new matrix of T, namely B, is given by $B = S^{-1}AS = \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix}$.

18. By Fact 3.4.4, $A = SBS^{-1} = \begin{bmatrix} 3 & 5 \\ 5 & 8 \end{bmatrix}\begin{bmatrix} 1 & 9 \\ 9 & 7 \end{bmatrix}\begin{bmatrix} 3 & 5 \\ 5 & 8 \end{bmatrix}^{-1} = \begin{bmatrix} -74 & 54 \\ -111 & 82 \end{bmatrix}$.

29. By Fact 3.4.1, $\vec{x} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}[\vec{x}]_{\mathcal{B}}$ and $\vec{x} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}[\vec{x}]_{\mathcal{R}}$, so that $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}[\vec{x}]_{\mathcal{R}}$ and $[\vec{x}]_{\mathcal{R}} =$

$\underbrace{\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}^{-1}\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}}_{P}[\vec{x}]_{\mathcal{B}}$, i.e. $P = \begin{bmatrix} -\frac{1}{2} & 1 \\ \frac{1}{2} & 0 \end{bmatrix}$.

32. a. Consider a linear relation $c_1 A^2\vec{v} + c_2 A\vec{v} + c_3\vec{v} = \vec{0}$.

Multiplying A^2 with the vectors on both sides and using that $A^3\vec{v} = \vec{0}$ and $A^4\vec{v} = \vec{0}$, we find that $c_3 A^2\vec{v} = \vec{0}$ and therefore $c_3 = 0$, since $A^2\vec{v} \neq \vec{0}$.

Therefore, our relation simplifies to $c_1 A^2\vec{v} + c_2 A\vec{v} = \vec{0}$.

Multiplying A with the vectors on both sides we find that $c_2 A^2\vec{v} = \vec{0}$ and therefore $c_2 = 0$. Then $c_1 = 0$ as well. We have shown that there is only the trivial relation among the vectors $A^2\vec{v}, A\vec{v}$, and $\vec{v}$, so that these three vectors from a basis of $\mathbb{R}^3$, as claimed.

b. $T(A^2\vec{v}) = A^3\vec{v} = \vec{0}$ so $[T(A^2\vec{v})]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

$T(A\vec{v}) = A^2\vec{v}$ so $[T(A\vec{v})]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

$T(\vec{v}) = A\vec{v}$ so $[T(\vec{v})]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

Hence, by Fact 3.4.3, the desired matrix is $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

36. We seek a basis $\vec{v}_1 = \begin{bmatrix} x \\ z \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} y \\ t \end{bmatrix}$ such that the matrix $S = [\vec{v}_1 \ \vec{v}_2] = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$ satisfies the equation

$\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} x & y \\ z & t \end{bmatrix} = \begin{bmatrix} x & y \\ z & t \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}$. Solving the ensuing linear system gives $S = \begin{bmatrix} \frac{z}{2} & -t \\ z & t \end{bmatrix}$. We need to choose both z and t nonzero to make S invertible. For example, if we let $z = 2$ and $t = 1$, then $S = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$, so that $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

44. Suppose such a basis $\vec{v}_1, \vec{v}_2$ exists. If $B = [[T(\vec{v}_1)]_{\mathcal{B}} \ [T(\vec{v}_2)]_{\mathcal{B}}]$ is upper triangular, of the form $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$, then $[T(\vec{v}_1)]_{\mathcal{B}} = \begin{bmatrix} a \\ 0 \end{bmatrix}$, so that $T(\vec{v}_1) = a\vec{v}_1$, that is, $T(\vec{v}_1)$ is parallel to $\vec{v}_1$. But this is impossible, since T is a rotation through $\frac{\pi}{2}$.