

6.  $\alpha = \arccos \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|} = \arccos \frac{2 - 3 + 8 - 10}{\sqrt{10} \sqrt{54}} \approx 1.700$  (radians)

14. The horizontal components of  $\vec{F}_1$  and  $\vec{F}_2$  are  $-\|\vec{F}_1\| \sin \beta$  and  $\|\vec{F}_2\| \sin \alpha$ , respectively (the horizontal component of  $\vec{F}_3$  is zero).

Since the system is at rest, the horizontal components must add up to 0, so that  $-\|\vec{F}_1\| \sin \beta + \|\vec{F}_2\| \sin \alpha = 0$  or  $\|\vec{F}_1\| \sin \beta = \|\vec{F}_2\| \sin \alpha$  or  $\frac{\|\vec{F}_1\|}{\|\vec{F}_2\|} = \frac{\sin \alpha}{\sin \beta}$ .

To find  $\frac{\overline{EA}}{\overline{EB}}$ , note that  $\overline{EA} = \overline{ED} \tan \alpha$  and  $\overline{EB} = \overline{ED} \tan \beta$  so that  $\frac{\overline{EA}}{\overline{EB}} = \frac{\tan \alpha}{\tan \beta} = \frac{\sin \alpha}{\sin \beta} \cdot \frac{\cos \beta}{\cos \alpha} = \frac{\|\vec{F}_1\| \cos \beta}{\|\vec{F}_2\| \cos \alpha}$ . Since  $\alpha$  and  $\beta$  are two distinct acute angles, it follows that  $\frac{\overline{EA}}{\overline{EB}} \neq \frac{\|\vec{F}_1\|}{\|\vec{F}_2\|}$ , so that Leonardo was mistaken.

26. The two given vectors spanning the subspace are orthogonal, but they are not unit vectors: both have length 7. To obtain an orthonormal basis  $\vec{v}_1, \vec{v}_2$  of the subspace, we divide by 7:

$$\vec{v}_1 = \frac{1}{7} \begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix}, \vec{v}_2 = \frac{1}{7} \begin{bmatrix} 3 \\ -6 \\ 2 \end{bmatrix}.$$

Now we can use Fact 5.1.6, with  $\vec{x} = \begin{bmatrix} 49 \\ 49 \\ 49 \end{bmatrix}$ :

$$\text{proj}_V \vec{x} = (\vec{v}_1 \cdot \vec{x}) \vec{v}_1 + (\vec{v}_2 \cdot \vec{x}) \vec{v}_2 = 11 \begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix} - \begin{bmatrix} 3 \\ -6 \\ 2 \end{bmatrix} = \begin{bmatrix} 19 \\ 39 \\ 64 \end{bmatrix}.$$

28. Since the three given vectors in the subspace are orthogonal, we have the orthonormal basis

$$\vec{v}_1 = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_2 = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \vec{v}_3 = \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}.$$

Now we can use Fact 5.1.6, with  $\vec{x} = \vec{e}_1$ :  $\text{proj}_V \vec{x} = (\vec{v}_1 \cdot \vec{x}) \vec{v}_1 + (\vec{v}_2 \cdot \vec{x}) \vec{v}_2 + (\vec{v}_3 \cdot \vec{x}) \vec{v}_3 = \frac{1}{4} \begin{bmatrix} 3 \\ 1 \\ -1 \\ 1 \end{bmatrix}$ .

29. By the Pythagorean theorem (Fact 5.1.8),

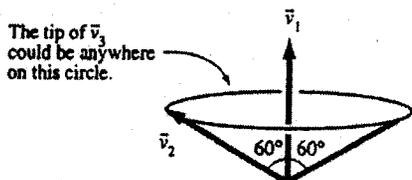
$$\begin{aligned} \|\vec{x}\|^2 &= \|7\vec{v}_1 - 3\vec{v}_2 + 2\vec{v}_3 + \vec{v}_4 - \vec{v}_5\|^2 \\ &= \|7\vec{v}_1\|^2 + \|3\vec{v}_2\|^2 + \|2\vec{v}_3\|^2 + \|\vec{v}_4\|^2 + \|\vec{v}_5\|^2 \\ &= 49 + 9 + 4 + 1 + 1 \\ &= 64, \text{ so that } \|\vec{x}\| = 8. \end{aligned}$$

36. Let  $\vec{x} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$  and  $\vec{y} = \begin{bmatrix} 0.2 \\ 0.3 \\ 0.5 \end{bmatrix}$ . It is required that  $\vec{x} \cdot \vec{y} = 0.2a + 0.3b + 0.5c = 76$ . Our goal is to minimize quantity  $\vec{x} \cdot \vec{x} = a^2 + b^2 + c^2$ . The Cauchy-Schwarz inequality (squared) tells us that  $(\vec{x} \cdot \vec{y})^2 \leq \|\vec{x}\|^2 \|\vec{y}\|^2$ , or  $76^2 \leq (a^2 + b^2 + c^2)(0.2^2 + 0.3^2 + 0.5^2)$  or  $a^2 + b^2 + c^2 \geq \frac{76^2}{0.38}$ . Quantity  $a^2 + b^2 + c^2$  is minimal when  $a^2 + b^2 + c^2 = \frac{76^2}{0.38}$ . This is the case when  $\vec{x} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0.2k \\ 0.3k \\ 0.5k \end{bmatrix}$  for some positive constant  $k$ . It is required that  $0.2a + 0.3b + 0.5c = (0.2)^2 k + (0.3)^2 k + (0.5)^2 k = 0.38k = 76$ , so that  $k = 200$ . Thus  $a = 40, b = 60, c = 100$ : The student must study 40 hours for the first exam, 60 hours for the second, and 100 hours for the third.

38. Since  $\vec{v}_1$  and  $\vec{v}_2$  are unit vectors, the condition  $\vec{v}_1 \cdot \vec{v}_2 = \|\vec{v}_1\| \|\vec{v}_2\| \cos(\alpha) = \cos(\alpha) = \frac{1}{2}$  implies that  $\vec{v}_1$  and  $\vec{v}_2$  enclose an angle of  $60^\circ$  ( $= \frac{\pi}{3}$ ). The vectors  $\vec{v}_1$  and  $\vec{v}_3$  enclose an angle of  $60^\circ$  as well.

In the case  $n = 2$  there are two possible scenarios: either  $\vec{v}_2 = \vec{v}_3$ , or  $\vec{v}_2$  and  $\vec{v}_3$  enclose an angle of  $120^\circ$ . Therefore, either  $\vec{v}_2 \cdot \vec{v}_3 = 1$  or  $\vec{v}_2 \cdot \vec{v}_3 = \cos(120^\circ) = -\frac{1}{2}$ .

In the case  $n = 3$ , the vectors  $\vec{v}_2$  and  $\vec{v}_3$  could enclose any angle between  $0^\circ$  (if  $\vec{v}_2 = \vec{v}_3$ ) and  $120^\circ$ , as illustrated in the figure below. We have  $-\frac{1}{2} \leq \vec{v}_2 \cdot \vec{v}_3 \leq 1$ .



$$\text{For example, consider } \vec{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} \left(\frac{\sqrt{3}}{2}\right) \cos \theta \\ \left(\frac{\sqrt{3}}{2}\right) \sin \theta \\ \frac{1}{2} \end{bmatrix}$$

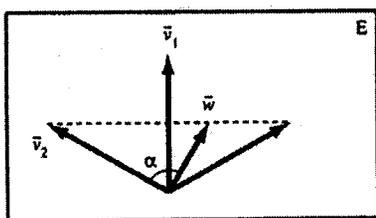
Note that  $\vec{v}_2 \cdot \vec{v}_3 = \left(\frac{3}{4}\right) \sin \theta + \frac{1}{4}$  could be anything between  $-\frac{1}{2}$  (when  $\sin \theta = -1$ ) and  $1$  (when  $\sin \theta = 1$ ), as claimed.

If  $n$  exceeds three, we can consider the orthogonal projection  $\vec{w}$  of  $\vec{v}_3$  onto the plane  $E$  spanned by  $\vec{v}_1$  and  $\vec{v}_2$ .

Since  $\text{proj}_{\vec{v}_1} \vec{w} = (\vec{v}_1 \cdot \vec{w}) \vec{v}_1 = \frac{1}{2} \vec{v}_1$ , and since  $\|\vec{w}\| \leq \|\vec{v}_3\| = 1$ , (by Fact 5.1.9), the tip of  $\vec{w}$  will be on the line segment sketched below. Note that  $\alpha$  is between  $0^\circ$  and  $120^\circ$ , so that  $\cos \alpha$  is between  $-\frac{1}{2}$  and  $1$ .

Therefore,  $\vec{v}_2 \cdot \vec{v}_3 = \vec{v}_2 \cdot \vec{w} = \|\vec{w}\| \cos \alpha$  is between  $-\frac{1}{2}$  and  $1$ .

This implies that  $\angle(\vec{v}_2, \vec{v}_3)$  is between  $0^\circ$  and  $120^\circ$  as well. To see that all these values are attained, add  $(n-3)$  zeros to the three vectors  $\vec{v}_1, \vec{v}_2, \vec{v}_3$  in  $\mathbb{R}^3$  given above.



$$4. \vec{w}_1 = \frac{1}{5} \begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix} \text{ and } \vec{w}_2 = \frac{1}{5} \begin{bmatrix} 3 \\ 0 \\ -4 \end{bmatrix} \text{ as in Exercise 3.}$$

$$\text{Since } \vec{v}_3 \text{ is orthogonal to } \vec{w}_1 \text{ and } \vec{w}_2, \vec{w}_3 = \frac{1}{\|\vec{v}_3\|} \vec{v}_3 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}.$$

$$14. \vec{w}_1 = \frac{1}{\|\vec{v}_1\|} \vec{v}_1 = \frac{1}{10} \begin{bmatrix} 1 \\ 7 \\ 1 \\ 7 \end{bmatrix}$$

$$\vec{w}_2 = \frac{\vec{v}_2 - (\vec{w}_1 \cdot \vec{v}_2) \vec{w}_1}{\text{length}} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\vec{w}_3 = \frac{\vec{v}_3 - (\vec{w}_1 \cdot \vec{v}_3) \vec{w}_1 - (\vec{w}_2 \cdot \vec{v}_3) \vec{w}_2}{\text{length}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$$

In Exercises 15–28, we will use the results of Exercises 1–14 (note that Exercise  $k$ , where  $k = 1, \dots, 14$ , gives the  $QR$  factorization of Exercise  $(k + 14)$ ). We can set  $Q = [\vec{w}_1 \dots \vec{w}_m]$ ; the entries of  $R$  are

$$r_{11} = \|\vec{v}_1\|$$

$$r_{22} = \|\vec{v}_2 - (\vec{w}_1 \cdot \vec{v}_2)\vec{w}_1\|$$

$$r_{33} = \|\vec{v}_3 - (\vec{w}_1 \cdot \vec{v}_3)\vec{w}_1 - (\vec{w}_2 \cdot \vec{v}_3)\vec{w}_2\|$$

$$r_{ij} = \vec{w}_i \cdot \vec{v}_j, \text{ where } i < j.$$

32. A basis of the plane is  $\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ ,  $\vec{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ .

Now apply the Gram-Schmidt process.

$$\vec{w}_1 = \frac{1}{\|\vec{v}_1\|} \vec{v}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$\vec{w}_2 = \frac{\vec{v}_2 - (\vec{w}_1 \cdot \vec{v}_2)\vec{w}_1}{\text{length}} = \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$$

Your solution may be different if you start with a different basis  $\vec{v}_1, \vec{v}_2$  of the plane.

34.  $\text{rref}(A) = \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \end{bmatrix}$

A basis of  $\ker(A)$  is  $\vec{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$ ,  $\vec{v}_2 = \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$ .

We apply the Gram-Schmidt process and obtain

$$\vec{w}_1 = \frac{1}{\|\vec{v}_1\|} \vec{v}_1 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \vec{w}_2 = \frac{\vec{v}_2 - (\vec{w}_1 \cdot \vec{v}_2)\vec{w}_1}{\text{length}} = \frac{1}{\sqrt{30}} \begin{bmatrix} 2 \\ -1 \\ -4 \\ 3 \end{bmatrix}$$

40. If  $\vec{v}_1, \dots, \vec{v}_n$  are the columns of  $A$ , then  $Q = \begin{bmatrix} \vec{v}_1 & \dots & \vec{v}_n \\ \|\vec{v}_1\| & \dots & \|\vec{v}_n\| \end{bmatrix}$  and  $R = \begin{bmatrix} \|\vec{v}_1\| & & 0 \\ & \dots & \\ 0 & & \|\vec{v}_n\| \end{bmatrix}$ .

(See Exercise 38 as an example.)

8. a. No! As a counterexample, consider  $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$  (see Exercise 4).

b. Yes! More generally, if  $A$  and  $B$  are  $n \times n$  matrices such that  $BA = I_n$ , then  $AB = I_n$ , by Fact 2.4.9c.

10. Since the first two columns are orthogonal to the third, we have  $c = d = 0$ . Then  $\begin{bmatrix} a & b \\ e & f \end{bmatrix}$  is an orthogonal  $2 \times 2$  matrix; By Exercise 9, the  $3 \times 3$  matrix  $A$  is either of the form  $A = \begin{bmatrix} \cos(\phi) & -\sin(\phi) & 0 \\ 0 & 0 & 1 \\ \sin(\phi) & \cos(\phi) & 0 \end{bmatrix}$

or  $A = \begin{bmatrix} \cos(\phi) & \sin(\phi) & 0 \\ 0 & 0 & 1 \\ \sin(\phi) & -\cos(\phi) & 0 \end{bmatrix}$ .

14. By Fact 5.3.9a,  $(A^T A)^T = A^T (A^T)^T = A^T A$ .  
The matrix  $AA^T$  is symmetric as well:  $(AA^T)^T = (A^T)^T A^T = AA^T$ .

18. a. The general form of a skew-symmetric  $3 \times 3$  matrix is  $A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$ , with

$$A^2 = \begin{bmatrix} -a^2 - b^2 & -bc & ac \\ -bc & -a^2 - c^2 & -ab \\ ac & -ab & -b^2 - c^2 \end{bmatrix}, \text{ a symmetric matrix.}$$

b. By Fact 5.3.9.a,  $(A^2)^T = (A^T)^2 = (-A)^2 = A^2$ , so that  $A^2$  is symmetric.

19. By Fact 5.3.10, the matrix of the projection is  $\vec{v}\vec{v}^T$ ; the  $ij$ th entry of this matrix is  $v_i v_j$ .

20. An orthonormal basis of  $W$  is  $\vec{w}_1 = \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \end{bmatrix}$ ,  $\vec{w}_2 = \begin{bmatrix} -0.1 \\ 0.7 \\ -0.7 \\ 0.1 \end{bmatrix}$  (see Exercise 5.2.9).

By Fact 5.3.10, the matrix of the projection onto  $W$  is  $AA^T$ , where  $A = [\vec{w}_1 \quad \vec{w}_2]$ .

$$AA^T = \frac{1}{100} \begin{bmatrix} 26 & 18 & 32 & 24 \\ 18 & 74 & -24 & 32 \\ 32 & -24 & 74 & 18 \\ 24 & 32 & 18 & 26 \end{bmatrix}$$

40. Using Fact 4.3.3, we find the matrix  $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$ .

44. a.  $\begin{bmatrix} A & -B^T \\ B & A^T \end{bmatrix} + \begin{bmatrix} C & -D^T \\ D & C^T \end{bmatrix} = \begin{bmatrix} A+C & -B^T - D^T \\ B+D & A^T + C^T \end{bmatrix} = \begin{bmatrix} A+C & -(B+D)^T \\ B+D & (A+C)^T \end{bmatrix}$  is of the required form.

b.  $k \begin{bmatrix} A & -B^T \\ B & A^T \end{bmatrix} = \begin{bmatrix} kA & -kB^T \\ kB & kA^T \end{bmatrix} = \begin{bmatrix} kA & -(kB)^T \\ kB & (kA)^T \end{bmatrix}$  is of the required form.

c. The general element of  $H$  is  $M = \begin{bmatrix} p & -q & -r & -s \\ q & p & s & -r \\ r & -s & p & q \\ s & r & -q & p \end{bmatrix}$ , with four arbitrary constants,  $r, s, p$ , and  $q$ .

Thus  $\dim(H) = 4$ ; use the strategy outlined on Page 155 to construct a basis.

d.  $\begin{bmatrix} A & -B^T \\ B & A^T \end{bmatrix} \begin{bmatrix} C & -D^T \\ D & C^T \end{bmatrix} = \begin{bmatrix} AC - B^T D & -AD^T - B^T C^T \\ BC + A^T D & -BD^T + A^T C^T \end{bmatrix} = \begin{bmatrix} AC - B^T D & -(BC + A^T D)^T \\ BC + A^T D & (AC - B^T D)^T \end{bmatrix}$  is of the required form.

Note that  $A, B, C, D$ , and their transposes are rotation-dilation matrices, so that they all commute.

e.  $\begin{bmatrix} A & -B^T \\ B & A^T \end{bmatrix}^T = \begin{bmatrix} A^T & B^T \\ -B & (A^T)^T \end{bmatrix}$  is of the required form.

f. Note that the columns  $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$  or  $M$  are orthogonal, and they all have length  $\sqrt{p^2 + q^2 + r^2 + s^2}$ . Now  $M^T M$  is the  $4 \times 4$  matrix whose  $ij$ th entry is  $\vec{v}_i \cdot \vec{v}_j$ , so that  $M^T M = (p^2 + q^2 + r^2 + s^2)I_4$ .

g. If  $M \neq 0$ , then  $k = p^2 + q^2 + r^2 + s^2 > 0$ , and  $\left(\frac{1}{k}M^T\right)M = I_4$ , so that  $M$  is invertible, with

$$M^{-1} = \frac{1}{p^2 + q^2 + r^2 + s^2} M^T.$$

By parts b and e,  $M^{-1}$  is in  $H$  as well.

h. No!  $A = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$  and  $B = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$  do not commute ( $AB = -BA$ ).