10. a. If $\vec{x}$ is an arbitrary solution of the system $A\vec{x} = \vec{b}$, let $\vec{x}_h = \text{proj}_V \vec{x}$, where $V = \ker(A)$, and $\vec{x}_0 = \vec{x} - \text{proj}_V \vec{x}$. Note that $\vec{b} = A\vec{x} = A(\vec{x}_h + \vec{x}_0) = A\vec{x}_h + A\vec{x}_0 = A\vec{x}_0$, since $\vec{x}_h$ is in $\ker(A)$.

b. If $\vec{x}_0$ and $\vec{x}_1$ are two solutions of the system $A\vec{x} = \vec{b}$, both from $(\ker A)^\perp$, then $\vec{x}_1 - \vec{x}_0$ is in the subspace $(\ker A)^\perp$ as well. Also, $A(\vec{x}_1 - \vec{x}_0) = A\vec{x}_1 - A\vec{x}_0 = \vec{b} - \vec{b} = \vec{0}$, so that $\vec{x}_1 - \vec{x}_0$ is in $\ker(A)$. By Fact 5.4.2c, it follows that $\vec{x}_1 - \vec{x}_0 = \vec{0}$, or $\vec{x}_1 = \vec{x}_0$, as claimed.

c. Write $\vec{x}_1 = \vec{x}_h + \vec{x}_0$ as in part a; note that $\vec{x}_h$ is orthogonal to $\vec{x}_0$. The claim now follows from the Pythagorean Theorem (Fact 5.1.8).

20. Using Fact 5.4.7, we find $\vec{x}^* = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ and $\vec{b} - A\vec{x}^* = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$.

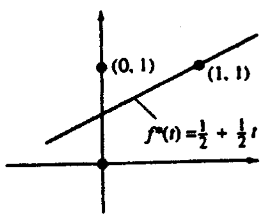
Note that $\vec{b} - A\vec{x}^*$ is perpendicular to the two columns of A.

22. Using Fact 5.4.7, we find $\vec{x}^* = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ and $\vec{b} - A\vec{x}^* = \vec{0}$. This system is in fact consistent and $\vec{x}^*$ is the exact solution; the error $\|\vec{b} - A\vec{x}^*\|$ is 0.

30. We attempt to solve the system

$$\begin{aligned} c_0 + 0c_1 &= 0 \\ c_0 + 0c_1 &= 1 \\ c_0 + 1c_1 &= 1 \end{aligned}, \text{ or } \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

This system cannot be solved exactly; the least-squares solution is $\begin{bmatrix} c_0^* \\ c_1^* \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$. The line that fits the data points best is $f^*(t) = \frac{1}{2} + \frac{1}{2}t$.



The line goes through the point (1, 1) and "splits the difference" between (0, 0) and (0, 1).

36. We want $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ such that

$$a + b\sin\left(\frac{2\pi}{365}28\right) + c\cos\left(\frac{2\pi}{365}28\right) = 10$$

$$a + b\sin\left(\frac{2\pi}{365}77\right) + c\cos\left(\frac{2\pi}{365}77\right) = 12$$

$$a + b\sin\left(\frac{2\pi}{365}124\right) + c\cos\left(\frac{2\pi}{365}124\right) = 14$$

$$a + b\sin\left(\frac{2\pi}{365}168\right) + c\cos\left(\frac{2\pi}{365}168\right) = 15.$$

Using $A = \begin{bmatrix} 1 & \sin\left(\frac{2\pi}{365}28\right) & \cos\left(\frac{2\pi}{365}28\right) \\ 1 & \sin\left(\frac{2\pi}{365}77\right) & \cos\left(\frac{2\pi}{365}77\right) \\ 1 & \sin\left(\frac{2\pi}{365}124\right) & \cos\left(\frac{2\pi}{365}124\right) \\ 1 & \sin\left(\frac{2\pi}{365}168\right) & \cos\left(\frac{2\pi}{365}168\right) \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 10 \\ 12 \\ 14 \\ 15 \end{bmatrix}$ we compute $\begin{bmatrix} a \\ b \\ c \end{bmatrix}^* = (A^TA)^{-1}A^T\vec{b} \approx$

$\begin{bmatrix} 12.25 \\ 0.394 \\ -2.726 \end{bmatrix}$ and $f^*(t) \approx 12.25 + 0.394\sin\left(\frac{2\pi}{365}t\right) - 2.726\cos\left(\frac{2\pi}{365}t\right)$.

38. We want $\begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix}$ such that

$$\begin{aligned} 110 &= c_0 + 2c_1 + c_2 \\ 180 &= c_0 + 12c_1 + 0c_2 \\ 120 &= c_0 + 5c_1 + c_2 \\ 160 &= c_0 + 11c_1 + c_2 \\ 160 &= c_0 + 6c_1 + 0c_2 \end{aligned} \quad \text{or} \quad \begin{bmatrix} 1 & 2 & 1 \\ 1 & 12 & 0 \\ 1 & 5 & 1 \\ 1 & 11 & 1 \\ 1 & 6 & 0 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 110 \\ 180 \\ 120 \\ 160 \\ 160 \end{bmatrix}.$$

The least-squares solution is $\begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix}^* = \begin{bmatrix} 125 \\ 5 \\ -25 \end{bmatrix}$, so that $w^* = 125 + 5h - 25g$.

For a general population, we expect c_0 and c_1 to be positive, since c_0 gives the weight of a 5′ male, and increased height should contribute positively to the weight. We expect c_2 to be negative, since females tend to be lighter than males of equal height.

40. First we look for $\begin{bmatrix} c_0 \\ c_1 \end{bmatrix}$ such that $\log D = c_0 + c_1 \log a$.

Proceeding as in Exercise 39, we get $\begin{bmatrix} c_0 \\ c_1 \end{bmatrix}^* \approx \begin{bmatrix} 0 \\ 1.5 \end{bmatrix}$, i.e. $\log D \approx 1.5 \log a$, hence $D \approx 10^{1.5 \log a} = a^{1.5}$.

Note that the formula $D = a^{1.5}$ is Kepler's third law of planetary motion.

41. We ... $\begin{bmatrix} c_0 \end{bmatrix}$

52. The pattern containing all the 1000's has 4 inversions so it contributes $(1000)^5 = 10^{15}$ to the value of the determinant. There are $5! - 1 = 119$ other patterns; the product associated with each of these patterns is less than $1000^3 \cdot 9^2 < 10^{11}$. Therefore we can say that $\det(A) > 0$.

6.2

4. $\det \begin{bmatrix} 0 & 2 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 & 2 \\ 0 & 5 & 3 & 9 & 9 \\ 0 & 7 & 4 & 0 & 1 \\ 3 & 9 & 5 & 4 & 8 \end{bmatrix} = -324$

14. This determinant is 0, since the first row is twice the last.

15. If a square matrix A has two equal columns, then its columns are linearly dependent, hence A is not invertible, and $\det(A) = 0$.

16. a. $f(t) = \det \begin{bmatrix} 1 & 1 & 1 \\ a & b & t \\ a^2 & b^2 & t^2 \end{bmatrix} = (ab^2 - a^2b) + (a^2 - b^2)t + (b-a)t^2$ so $f(t)$ is a quadratic function of t.
The coefficient of t^2 is $(b - a)$.

b. In the cases $t = a$ and $t = b$ the matrix has two identical columns; compare with Exercise 15. It follows that $f(t) = k(t-a)(t-b)$ with $k =$ coefficient of $t^2 = (b - a)$.

c. The matrix is invertible for the values of t for which $f(t) \neq 0$, i.e., for $t \neq a, t \neq b$.

26. By Exercise 25, $\det(A^TA) = [\det(A)]^2$. Since A is orthogonal, $A^TA = I_n$ so that $1 = \det(I_n) = \det(A^TA) = [\det(A)]^2$ and $\det(A) = \pm 1$.

34. a. $D_1 = 2, D_2 = 3, D_3 = 4, D_4 = 5$

b. $D_n = 2D_{n-1} - D_{n-2}$ (expand along the first column to see this)

c. $D_n = n + 1$ (by induction)

37. Expand along the first column, realizing that all but the first contribution is zero, since the other minors will have two equal rows. Therefore, $\det(P_n) = \det(P_{n-1})$.
Since $\det(P_1) = 1$ we can conclude that $\det(P_n) = 1$, for all n.

7. We know $A\vec{v} = \lambda\vec{v}$ so $(\lambda I_n - A)\vec{v} = \lambda I_n\vec{v} - A\vec{v} = \lambda\vec{v} - \lambda\vec{v} = \vec{0}$ so a nonzero vector $\vec{v}$ is in the kernel of $(\lambda I_n - A)$ so $\ker(\lambda I_n - A) \neq \{\vec{0}\}$ and $\lambda I_n - A$ is not invertible.

8. We want all $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ hence $\begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$, i.e. the desired matrices must have the form $\begin{bmatrix} 5 & b \\ 0 & d \end{bmatrix}$.

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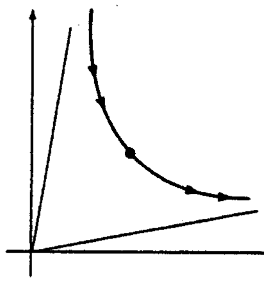
20. Any nonzero vector along the $\vec{e}_3$-axis is unchanged, hence is an eigenvector with eigenvalue 1. No other (real) eigenvalues can be found.

24.



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34. We want A such that $A\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 15 \\ 5 \end{bmatrix}$ and $A\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 10 \\ 20 \end{bmatrix}$, i.e. $A\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 15 & 10 \\ 5 & 20 \end{bmatrix}$, so

$A = \begin{bmatrix} 15 & 10 \\ 5 & 20 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 4 & 3 \\ -2 & 11 \end{bmatrix}$.

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36. a. $\begin{bmatrix} 0.978 & -0.006 \\ 0.004 & 0.992 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -0.99 \\ 1.98 \end{bmatrix} = 0.99\begin{bmatrix} -1 \\ 2 \end{bmatrix}$; and $\begin{bmatrix} 0.978 & -0.006 \\ 0.004 & 0.992 \end{bmatrix}\begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 2.94 \\ -0.98 \end{bmatrix} = 0.98\begin{bmatrix} 3 \\ -1 \end{bmatrix}$. The eigenvalues are $\lambda_1 = 0.99$ and $\lambda_2 = 0.98$.

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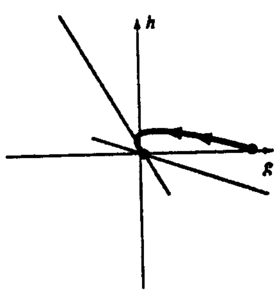
b. $\vec{x}_0 = \begin{bmatrix} g_0 \\ h_0 \end{bmatrix} = \begin{bmatrix} 100 \\ 0 \end{bmatrix} = 20\begin{bmatrix} -1 \\ 2 \end{bmatrix} + 40\begin{bmatrix} 3 \\ -1 \end{bmatrix}$ so $\vec{x}(t) = 20(0.99)^t\begin{bmatrix} -1 \\ 2 \end{bmatrix} + 40(0.98)^t\begin{bmatrix} 3 \\ -1 \end{bmatrix}$, hence $g(t) = -20(0.99)^t + 120(0.98)^t$ and $h(t) = 40(0.99)^t - 40(0.98)^t$.



$h(t)$ first rises, then falls back to zero. $g(t)$ falls a little below zero, then goes back up to zero.

c. We set $g(t) = -20(0.99)^t + 120(0.98)^t = 0$.
Solving for t we get that $g(t) = 0$ for $t \approx 176$ minutes. (After $t = 176$, $g(t) < 0$).

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38. a. We are given that
$n(t+1) = 2a(t)$
$a(t+1) = n(t) + a(t)$,
so that the matrix is $A = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$.

b. $A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $A\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} = -\begin{bmatrix} 2 \\ -1 \end{bmatrix}$, hence 2 and -1 are the eigenvalues associated with $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ respectively.

c. We are given $\vec{x}_0 = \begin{bmatrix} n_0 \\ a_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ so $\vec{x}_0 = \frac{1}{3}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$, and $\vec{x}(t) = \frac{1}{3}2^t\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{3}(-1)^t\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ (by Fact 7.1.3), hence $n(t) = \frac{1}{3}2^t + \frac{2}{3}(-1)^t$ and $a(t) = \frac{1}{3}2^t - \frac{1}{3}(-1)^t$.