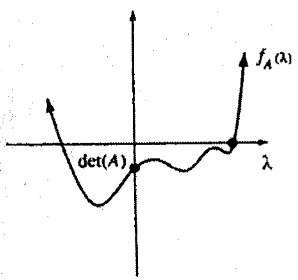
12. $\lambda_1 = \lambda_2 = \lambda_3 = 1$, $E_1 = \text{span}\begin{bmatrix}1\\0\\0\end{bmatrix}$, no eigenbasis

20. For $\lambda_1 = 1$, $E_1 = \ker\begin{bmatrix}0 & a & b\\0 & 0 & c\\0 & 0 & 1\end{bmatrix} = \ker\begin{bmatrix}0 & a & 0\\0 & 0 & 1\\0 & 0 & 0\end{bmatrix}$ so if $a = 0$ then E_1 is 2-dimensional, otherwise it is 1-dimensional.

For $\lambda_2 = 2$, $E_2 = \ker\begin{bmatrix}1 & -a & -b\\0 & 1 & -c\\0 & 0 & 0\end{bmatrix}$ so E_2 is 1-dimensional.

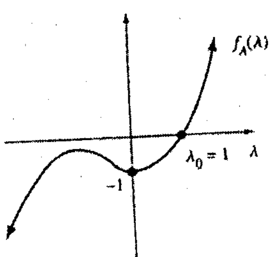
Hence, there is an eigenbasis if $a = 0$.

26. Note that $f_A(0) = \det(0I_6 - A) = \det(-A) = (-1)^6\det(A) = \det(A)$ is negative. Since $\lim_{\lambda\to\infty} f_A(\lambda) = \infty$, there must be a positive root, by the Intermediate Value Theorem (see Exercise 2.2.47c). Therefore, the matrix A has a positive eigenvalue.



36. No, since the two matrices have different traces (see Fact 7.3.8.d)

38. Note that $f_A(0) = \det(0I_3 - A) = \det(-A) = (-1)^3\det(A) = -1$. Since $\lim_{\lambda\to\infty} f_A(\lambda) = \infty$, the polynomial $f_A(\lambda)$ must have a positive root λ_0, by the Intermediate Value Theorem. In other words, the matrix A will have a positive eigenvalue λ_0. Since A is orthogonal, this eigenvalue λ_0 will be 1, by Fact 7.1.2. This means that there is a nonzero vector $\vec{v}$ in $\mathbb{R}^3$ such that $A\vec{v} = 1\vec{v} = \vec{v}$, as claimed.



44. a. $a_{11} = 0.7$ means that only 70% of the pollutant present in Lake Silvaplana at a given time is still there a week later; some is carried down to Lake Sils by the river Inn, and some is absorbed or evaporates.

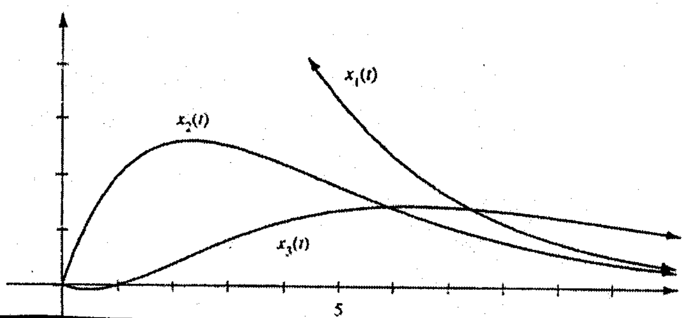
The other diagonal entries can be interpreted analogously. $a_{21} = 0.1$ means that 10% of the pollutant present in Lake Silvaplana at any given time can be found in Lake Sils a week later, carried down by the river Inn. The significance of the coefficient $a_{32} = 0.2$ is analogous; $a_{31} = 0$ means that no pollutant is carried down from Lake Silvaplana to Lake St. Moritz in just one week.

The matrix is lower triangular since no pollutant is carried from Lake Sils to Lake Silvaplana, for example (the river Inn flows the other way).

b. The eigenvalues of A are 0.8, 0.6, 0.7 with corresponding eigenvectors $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$.

$\vec{x}(0) = \begin{bmatrix} 100 \\ 0 \\ 0 \end{bmatrix} = 100 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - 100 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + 100 \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$ so $\vec{x}(t) = 100(0.8)^t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - 100(0.6)^t \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} +$ $100(0.7)^t \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$ or

$x_1(t) = 100(0.7)^t$
$x_2(t) = 100(0.7)^t - 100(0.6)^t$
$x_3(t) = 100(0.8)^t + 100(0.6)^t - 200(0.7)^t$



Using calculus, we find that the function $x_2(t) = 100(0.7)^t - 100(0.6)^t$ reaches its maximum at $t \approx 2.33$. Keep in mind, however, that our model holds for integer t only.

47. a. If $\vec{x}(t) = \begin{bmatrix} r(t) \\ p(t) \\ w(t) \end{bmatrix}$, then $\vec{x}(t+1) = A\vec{x}(t)$ with $A = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$.

The eigenvalues of A are $0, \frac{1}{2}, 1$ with eigenvectors $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

Since $\vec{x}(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \frac{1}{4} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $\vec{x}(t) = \frac{1}{2}\left(\frac{1}{2}\right)^t \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \frac{1}{4} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ for $t > 0$.

b. As $t \to \infty$ the ratio is $1 : 2 : 1$ (since the first term of $\vec{x}(t)$ drops out).

48. a. We are told that

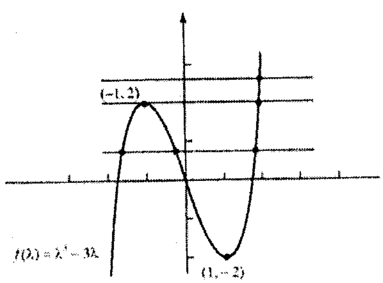
12. Diagonalizable. The eigenvalues are 2,1,1, with associated eigenvectors $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\1\end{bmatrix}$. If we let

$$S = \begin{bmatrix}1 & 0 & -1\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}, \text{ then } S^{-1}AS = D = \begin{bmatrix}2 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}.$$

16. Diagonalizable. The eigenvalues are 3,2,1, with associated eigenvectors $\begin{bmatrix}2\\0\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}$. If we let

$$S = \begin{bmatrix}2 & 1 & 0\\0 & 0 & 1\\1 & 1 & 0\end{bmatrix}, \text{ then } S^{-1}AS = D = \begin{bmatrix}3 & 0 & 0\\0 & 2 & 0\\0 & 0 & 1\end{bmatrix}.$$

30. First we observe that all the eigenspaces of $A = \begin{bmatrix}0 & 0 & a\\1 & 0 & 3\\0 & 1 & 0\end{bmatrix}$ are one-dimensional, regardless of the value of a, since rref$(\lambda I_3 - A)$ is of the form $\begin{bmatrix}1 & 0 & *\\0 & 1 & *\\0 & 0 & *\end{bmatrix}$ for all λ. Thus A is diagonalizable if and only if there are three distinct real eigenvalues. The characteristic polynomial of A is $\lambda^3 - 3\lambda - a$. Thus the eigenvalues of A are the solutions of the equation $\lambda^3 - 3\lambda = a$. See the accompanying graph of the function $f(\lambda) = \lambda^3 - 3\lambda$; using calculus, we find the local maximum $f(-1) = 2$ and the local minimum $f(1) = -2$. To count the distinct eigenvalues of A, we have to examine how many times the horizontal line $y = a$ intersects the graph of $f(\lambda)$. The answer is three if $|a| < 2$, two if $a = \pm 2$, and one if $|a| > 2$. Thus A is diagonalizable if and only if $|a| < 2$, that is, $-2 < a < 2$.



32. The eigenvalues of $A = \begin{bmatrix}4 & -2\\1 & 1\end{bmatrix}$ are 3 and 2, with associated eigenvectors $\begin{bmatrix}2\\1\end{bmatrix}$ and $\begin{bmatrix}1\\1\end{bmatrix}$. If we let

$S = \begin{bmatrix}2 & 1\\1 & 1\end{bmatrix}$, then $S^{-1}AS = D = \begin{bmatrix}3 & 0\\0 & 2\end{bmatrix}$. Thus

$$A = SDS^{-1} \text{ and } A^t = SD^tS^{-1} = \begin{bmatrix}2 & 1\\1 & 1\end{bmatrix}\begin{bmatrix}3^t & 0\\0 & 2^t\end{bmatrix}\begin{bmatrix}1 & -1\\-1 & 2\end{bmatrix} = \begin{bmatrix}2(3^t) - 2^t & 2^{t+1} - 2(3^t)\\3^t - 2^t & 2^{t+1} - 3^t\end{bmatrix}$$

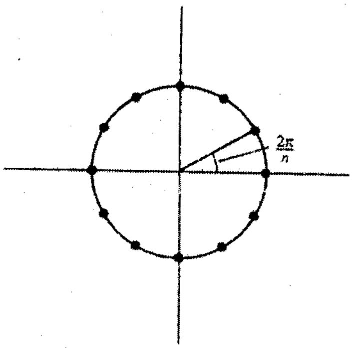
38. No. As a counterexample, consider $A = \begin{bmatrix}2 & 0\\0 & 2\end{bmatrix}$ and $B = \begin{bmatrix}2 & 1\\0 & 2\end{bmatrix}$.

54. Note that $A^2 = 0$, but $B^2 \neq 0$. Since A^2 fails to be similar to B^2, matrix A isn't similar to B (see Example 7 of Section 3.4).

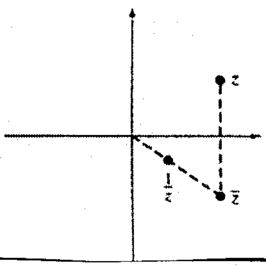
58. Let $B_i = A - \lambda_i I_n$; note that B_i and B_j commute for any two indices i and j. If $\vec{v}$ is an eigenvector of A with eigenvalue λ_i, then $B_i\vec{v} = \vec{0}$ and $B_1B_2\dots B_i\dots B_m\vec{v} = B_1\dots B_{i-1}B_{i+1}\dots B_mB_i\vec{v} = \vec{0}$. Since A is diagonalizable, any vector $\vec{x}$ in $\mathbb{R}^n$ can be written as a linear combination of eigenvectors, so that $B_1B_2\dots B_m\vec{x} = \vec{0}$ and therefore $B_1B_2\dots B_m = 0$, as claimed.

3. If $z = r(\cos\phi + i\sin\phi)$, then $z^n = r^n(\cos(n\phi) + i\sin(n\phi))$.

$z^n = 1$ if $r = 1, \cos(n\phi) = 1, \sin(n\phi) = 0$ so $n\phi = 2k\pi$ for an integer k, and $\phi = \dfrac{2k\pi}{n}$,
i.e. $z = \cos\left(\dfrac{2k\pi}{n}\right) + i\sin\left(\dfrac{2k\pi}{n}\right)$, $k = 0, 1, 2, \ldots, n-1$.



6. If we have $z = r(\cos\phi + i\sin\phi)$ then $\dfrac{1}{z}$ must have the property that $z \cdot \dfrac{1}{z} = 1 = \cos 0 + i\sin 0$
i.e. $|z| \cdot \left|\dfrac{1}{z}\right| = 1$ and $\arg\left(z \cdot \dfrac{1}{z}\right) = \arg(z) + \arg\left(\dfrac{1}{z}\right) = 0$ so $\dfrac{1}{z} = \dfrac{1}{r}(\cos(-\phi) + i\sin(-\phi)) =$
$\dfrac{1}{r}(\cos\phi - i\sin\phi)$ (since cosine is even, sine odd). Hence $\dfrac{1}{z}$ is a real scalar multiple of $\bar{z}$.



24. $f_A(\lambda) = \lambda^3 - 3\lambda^2 + 7\lambda - 5$ so $\lambda_1 = 1, \lambda_{2,3} = 1 \pm 2i$. (See Exercise 11.)

25. $f_A(\lambda) = \lambda^4 - 1 = (\lambda^2 - 1)(\lambda^2 + 1) = (\lambda - 1)(\lambda + 1)(\lambda - i)(\lambda + i)$ so $\lambda_{1,2} = \pm 1$ and $\lambda_{3,4} = \pm i$

26. $f_A(\lambda) = (\lambda^2 - 2\lambda + 2)(\lambda^2 - 2\lambda) = (\lambda^2 - 2\lambda + 2)(\lambda - 2)\lambda = 0$, so $\lambda_{1,2} = 1 \pm i, \lambda_3 = 2, \lambda_4 = 0$.

30. a. The ith entry of $A\vec{x}$ is $\sum_{k=1}^{n} a_{ik}x_k$, so that the sum of all the entries of $A\vec{x}$ is

$$\sum_{i=1}^{n}\sum_{k=1}^{n} a_{ik}x_k = \sum_{k=1}^{n}\sum_{i=1}^{n} a_{ik}x_k = \sum_{k=1}^{n}\Big(\underbrace{\sum_{i=1}^{n} a_{ik}}_{1}\Big)x_k = \sum_{k=1}^{n} x_k = 1.$$

b. As we do some computer experiments, A^t appears to approach a matrix with identical columns, with column sum 1. Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ be an eigenbasis with $\lambda_1 = 1$ and $|\lambda_j| < 1$ for $j = 2, \ldots, n$. For a *fixed* i, write $\vec{e}_i = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$, so that ($i$th column of A^t) $= A^t\vec{e}_i = c_1\vec{v}_1 + [c_2\lambda_2^t\vec{v}_2 + \cdots + c_n\lambda_n^t\vec{v}_n]$.
(The term in *square* brackets goes to zero as t goes to infinity.)
Therefore, $\lim_{t\to\infty}$ (ith column of A^t) $= \lim_{t\to\infty}(A^t\vec{e}_i) = c_1\vec{v}_1$.
Furthermore, the entries of $A^t\vec{e}_i$ add up to 1, for all t, by part a. Therefore, the same is true for the limit (since the limit of a sum is the sum of the limits).
It follows that $\lim_{t\to\infty}(A^t)$ exists and has identical columns, with column sum 1, as claimed.

32. a. $\vec{x}(t) = \begin{bmatrix} a(t) \\ m(t) \\ s(t) \end{bmatrix} = \begin{bmatrix} 0.6a(t) + 0.1m(t) + 0.5s(t) \\ 0.2a(t) + 0.7m(t) + 0.1s(t) \\ 0.2a(t) + 0.2m(t) + 0.4s(t) \end{bmatrix}$ so $A = \begin{bmatrix} 0.6 & 0.1 & 0.5 \\ 0.2 & 0.7 & 0.1 \\ 0.2 & 0.2 & 0.4 \end{bmatrix}$.

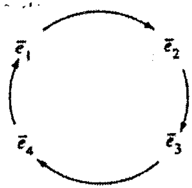
Note that A is a regular transition matrix.

b. By Exercise 30, $\lim_{t\to\infty}(A^t) = [\vec{v}\,\vec{v}\,\vec{v}]$, where $\vec{v}$ is the unique eigenvector of A with eigenvalue 1 and column sum 1. We find that $\vec{v} = \begin{bmatrix} 0.4 \\ 0.35 \\ 0.25 \end{bmatrix}$.

Now $\lim_{t\to\infty} \vec{x}(t) = \lim_{t\to\infty}(A^t\vec{x}_0) = \left(\lim_{t\to\infty} A^t\right)\vec{x}_0 = [\vec{v}\,\vec{v}\,\vec{v}]\vec{x}_0 = \vec{v}$, since the components of $\vec{x}_0$ add up to 1. The market shares approach 40%, 35%, and 25%, respectively, regardless of the initial shares.

38. a. $C_4^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, C_4^3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}, C_4^4 = I_4$, then $C_4^{4+k} = C_4^k$.

The diagram below illustrates how C_4 acts on the basis vectors $\vec{e}_i$:



b. The eigenvalues are $\lambda_1 = 1, \lambda_2 = -1, \lambda_3 = i$, and $\lambda_4 = -i$, and for each eigenvalue λ_k, $\vec{v}_k = \begin{bmatrix} \lambda_k^3 \\ \lambda_k^2 \\ \lambda_k \\ 1 \end{bmatrix}$ is an associated eigenvector.