Sooth functions $f(x)$ on $[-\pi, \pi]$ form a linear space X. There is an **inner product** in X defined by

$$\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)g(x)\, dx$$

It allows to define angles, length, projections etc in the space X as we did in finite dimensions.

THE FOURIER BASIS.

THEOREM. The functions $\{\cos(nx), \sin(nx), 1/\sqrt{2}\,\}$ form an orthonormal basis in X.

Proof. To check linear independence we have to evaluate a few integrals. For all $n, m \geq 1$, with $n \neq m$ we have

$$\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle = 1$$
$$\langle \cos(nx), \cos(nx) \rangle = 1, \langle \cos(nx), \cos(mx) \rangle = 0$$
$$\langle \sin(nx), \sin(nx) \rangle = 1, \langle \sin(nx), \sin(mx) \rangle = 0$$
$$\langle \sin(nx), \cos(mx) \rangle = 0$$
$$\langle \sin(nx), 1/\sqrt{2} \rangle = 0$$
$$\langle \cos(nx), 1/\sqrt{2} \rangle = 0$$

To verify the above integrals in the homework, the following trigonometric identities are useful:

$$2\cos(nx)\cos(my) = \cos(nx - my) + \cos(nx + my)$$
$$2\sin(nx)\sin(my) = \cos(nx - my) - \cos(nx + my)$$
$$2\sin(nx)\cos(my) = \sin(nx + my) + \sin(nx - my)$$

FOURIER COEFFICIENTS. The **Fourier coefficients** of f are defined as

$$a_0 = \langle f, 1/\sqrt{2} \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)/\sqrt{2}\, dx$$

$$a_n = \langle f, \cos(nt) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\cos(nx)\, dx$$

$$b_n = \langle f, \sin(nt) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\sin(nx)\, dx$$

FOURIER SERIES. The **Fourier representation** of a smooth function f is the identity

$$f(x) = \frac{a_0}{\sqrt{2}} + \sum_{k=1}^{\infty} a_k \cos(kx) + \sum_{k=1}^{\infty} b_k \sin(kx)$$

We take it for granted that the series converges and that the identity holds for all x.

ODD AND EVEN FUNCTIONS. Here is some advise which can save time when computing Fourier series:

If f is odd: $f(x) = -f(-x)$, then f has a sin series.
If f is even: $f(x) = -f(-x)$, then f has a cos series.
If you integrate an odd function over $[-\pi, \pi]$ you get 0.
The product of two odd functions is even, the product between an even and an odd function is odd.

EXAMPLE 1. Let $f(x) = x$ on $[-\pi, \pi]$. This is an odd function $(f(-x) + f(x) = 0)$ so that it has a sin series: with $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x\sin(nx)\, dx = \frac{-1}{\pi}(x\cos(nx)/n + \sin(nx)/n^2|_{-\pi}^{\pi}) = 2(-1)^{n+1}/n$, we get $x = \sum_{n=1}^{\infty} 2\frac{(-1)^{n+1}}{n} \sin(nx)$. For example

$$\frac{\pi}{2} = 2(\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}...)$$

is a **formula of Leibnitz**.

EXAMPLE 2. Let $f(x) = \cos(x) + 1/7\cos(5x)$. This **trigonometric polynomial** is already the Fourier series. The nonzero coefficients are $a_1 = 1, a_5 = 1/7$.

EXAMPLE 3. Let $f(x) = 1$ on $[-\pi/2, \pi/2]$ and $f(x) = 0$ else. This is an even function $f(-x) - f(x) = 0$ so that it has a cos series: with $a_0 = 1/(\sqrt{2}), a_n = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} 1\cos(nx)\, dx = \frac{\sin(nx)}{\pi n}|_{-\pi/2}^{\pi/2} = \frac{2(-1)^m}{\pi(2m+1)}$ if $n = 2m + 1$ is odd and 0 else. So, the series is

$$f(x) = \frac{1}{2} + \frac{2}{\pi}(\frac{\cos(x)}{1} - \frac{\cos(3x)}{3} + \frac{\cos(5x)}{5} - ...)$$

Remark. The function in Example 3 is not smooth but Fourier theory still works. What happened at the discontinuity point $\pi/2$? The Fourier series gives 0. Diplomatically, it has chosen the point in the middle of the limits from the right and the limit from the left.

FOURIER APPROXIMATION. For a smooth function f, the Fourier series of f converges to f. The Fourier coefficients are the coordinates of f in the Fourier basis.

The function $f_n(x) = \sum_{k=1}^{n} a_k \sin(kx)$ is called a **Fourier approximation** of f. The picture to the right shows an approximation of a piecewise continuous even function in example 3).
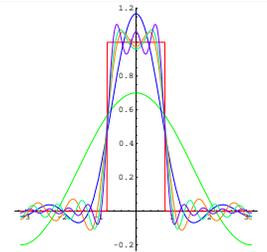
THE PARSEVAL EQUALITY. When evaluating the square of the length of f with the square of the length of the series, we get

$$||f||^2 = a_0^2 + \sum_{k=1}^{\infty} a_k^2 + b_k^2 .$$

EXAMPLE. We have seen in example 1 that $f(x) = x = 2(\sin(x) - \sin(2x)/2 + \sin(3x)/3 - \sin(4x)/4 + ...$ Because the Fourier coefficients are $a_k = 2(-1)^{k+1}/k$, we have $4(1 + 1/4 + 1/9 + ...) = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2\, dx = 2\pi^2/3$ and so

$$\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + ... = \frac{\pi^2}{6}$$

Isn't it fantastic that we can sum up the reciprocal squares? This formula has been obtained already by Leonard Euler, who celebrated his 300'th birthday two weeks ago, on April 15.

HOMEWORK: (this homework is due either Wednesday or Thursday. Remember that there is a lab due on Thursday or Friday).

1. Verify that the functions $\cos(nx), \sin(nx), 1/\sqrt{2}$ form an orthonormal family.

2. Find the Fourier series of the function $f(x) = |x|$.

3. Find the Fourier series of the function $\cos^2(x) + 5\sin(x) + 5$. You may find the double angle formula $\cos^2(x) = \frac{\cos(2x)+1}{2}$ useful.

4. Find the Fourier series of the function $f(x) = |\sin(x)|$.

5. In the previous problem 4) you should have gotten a series

$$f(x) = \frac{2}{\pi} - \frac{4}{\pi} \left(\frac{\cos(2x)}{2^2 - 1} + \frac{\cos(4x)}{4^2 - 1} + \frac{\cos(6x)}{6^2 - 1} + ... \right)$$

Use Parseval's identity to find the value of

$$\frac{1}{(2^2 - 1)^2} + \frac{1}{(4^2 - 1)^2} + \frac{1}{(6^2 - 1)^2} + \cdots$$

FUNCTIONS OF TWO VARIABLES. We consider functions $f(x,t)$ which are for fixed t a piecewise smooth function in x. Analogously as we studied the motion of a **vector** $\vec{v}(t)$, we are now interested in the motion of a **function** f in time t. While the governing equation for a vector was an ordinary differential equation $\dot{x} = Ax$ (ODE), the describing equation is now be a **partial differential equation** (PDE) $\dot{f} = T(f)$. The function $f(x,t)$ could denote the **temperature of a stick** at a position x at time t or the **displacement of a string** at the position x at time t. The motion of these dynamical systems will be easy to describe in the orthonormal Fourier basis $1/\sqrt{2}, \sin(nx), \cos(nx)$ treated in an earlier lecture.

PARTIAL DERIVATIVES. We write $f_x(x,t)$ and $f_t(x,t)$ for the **partial derivatives** with respect to x or t. The notation $f_{xx}(x,t)$ means that we differentiate twice with respect to x.

> Example: for $f(x,t) = \cos(x + 4t^2)$, we have
> - $f_x(x,t) = -\sin(x + 4t^2)$
> - $f_t(x,t) = -8t\sin(x + 4t^2)$.
> - $f_{xx}(x,t) = -\cos(x + 4t^2)$.

One also uses the notation $\frac{\partial f(x,y)}{\partial x}$ for the partial derivative with respect to x. Tired of all the "partial derivative signs", we always write $f_x(x,t)$ for the partial derivative with respect to x and $f_t(x,t)$ for the partial derivative with respect to t.

PARTIAL DIFFERENTIAL EQUATIONS. A partial differential equation is an equation for an unknown function $f(x,t)$ in which different partial derivatives occur.

> - $f_t(x,t) + f_x(x,t) = 0$ with $f(x,0) = \sin(x)$ has a solution $f(x,t) = \sin(x - t)$.
> - $f_{tt}(x,t) - f_{xx}(x,t) = 0$ with $f(x,0) = \sin(x)$ and $f_t(x,0) = 0$ has a solution $f(x,t) = (\sin(x - t) + \sin(x + t))/2$.

THE HEAT EQUATION. The temperature distribution $f(x,t)$ in a metal bar $[0,\pi]$ satisfies the **heat equation**

$$f_t(x,t) = \mu f_{xx}(x,t)$$

This partial differential equation tells that the rate of change of the temperature at x is proportional to the second space derivative of $f(x,t)$ at x. The function $f(x,t)$ is assumed to be zero at both ends of the bar and $f(x) = f(x,t)$ is a given initial temperature distribution. The constant μ depends on the heat conductivity properties of the material. Metals for example conduct heat well and would lead to a large μ.

REWRITING THE PROBLEM. We can write the problem as

$$\frac{d}{dt} f = \mu D^2 f$$

We will solve the problem in the same way as we solved linear differential equations:

$$\frac{d}{dt}\vec{x} = A\vec{x}$$

where A is a matrix - by diagonalization .

We use that the Fourier basis is just the diagonalization: $D^2 \cos(nx) = -n^2 \cos(nx)$ and $D^2 \sin(nx) = -n^2 \sin(nx)$ show that $\cos(nx)$ and $\sin(nx)$ are eigenfunctions to D^2 with eigenvalue n^2. By a symmetry trick, we can focus on sin-series from now on.

SOLVING THE HEAT EQUATION WITH FOURIER THEORY. The heat equation $f_t(x,t) = \mu f_{xx}(x,t)$ with smooth $f(x,0) = f(x), f(0,0) = f(\pi, t) = 0$ has the solution

$$f(x,t) = \sum_{n=1}^{\infty} b_n \sin(nx) e^{-n^2 \mu t} \qquad\qquad b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx$$

Proof: With the initial condition $f(x) = \sin(nx)$, we have the evolution $f(x,t) = e^{-\mu n^2 t}\sin(nx)$. If $f(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$ then $f(x,t) = \sum_{n=1}^{\infty} b_n e^{-\mu n^2 t}\sin(nx)$.
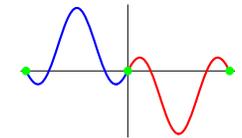
A SYMMETRY TRICK. Given a function f on the interval $[0,\pi]$ which is zero at 0 and π. It can be extended to an odd function on the doubled integral $[-\pi, \pi]$.

> The Fourier series of an odd function is a pure sin-series. The Fourier coefficients are $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx$.
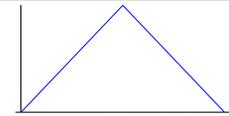
The function is given on $[0,\pi]$.
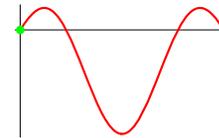
The odd symmetric extension on $[-\pi, \pi]$.

EXAMPLE. Assume the initial temperature distribution $f(x,0)$ is a sawtooth function which has slope $\pi/2$ on the interval $[0, \pi/2]$ and slope $-\pi/2$ on the interval $[\pi/2, \pi]$. We first compute the sin-Fourier coefficients of this function.

The sin-Fourier coefficients are $b_n = \frac{4}{n^2\pi}(-1)^{(n-1)/2}$ for odd n and 0 for even n. The solution is

$$f(x,t) = \sum_n^{\infty} b_n e^{-\mu n^2 t}\sin(nx) \ .$$

The exponential term containing the time makes the function $f(x,t)$ converge to 0: The body cools. The higher frequencies are damped faster: "smaller disturbances are smoothed out faster."

VISUALIZATION. We can plot the graph of the function $f(x,t)$ or slice this graph and plot the temperature distribution for different values of the time t.

$f(x,0)$ $f(x,1)$ $f(x,2)$ $f(x,3)$ $f(x,4)$
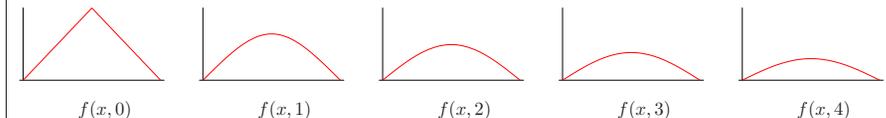
THE WAVE EQUATION. The height of a string $f(x,t)$ at time t and position x on $[0,\pi]$ satisfies the **wave equation**

$$f_{tt}(t,x) = c^2 f_{xx}(t,x)$$

where c is a constant. As we will see, c is the **speed** of the waves.

REWRITING THE PROBLEM. We can write the problem as

$$\frac{d^2}{dt^2}f = c^2 D^2 f$$

We will solve the problem in the same way as we solved

$$\frac{d^2}{dt^2}\vec{x} = A\vec{x}$$

If A is diagonal, then every basis vector x satisfies an equation of the form $\frac{d^2}{dt^2}x = -c^2 x$ which has the solution $x(t) = x(0)\cos(ct) + x(t)\sin(ct)/c$.

SOLVING THE WAVE EQUATION WITH FOURIER THEORY. The wave equation $f_{tt} = c^2 f_{xx}$ with $f(x,0) = f(x), f_t(x,0) = g(x), f(0,t) = f(\pi,t) = 0$ has the solution

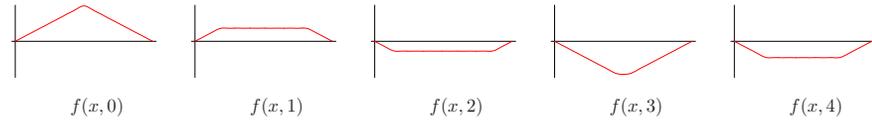$$f(x,t) = \sum_{n=1}^{\infty} a_n \sin(nx)\cos(nct) + \frac{b_n}{nc}\sin(nx)\sin(nct)$$

$$a_n = \frac{2}{\pi}\int_0^\pi f(x)\sin(nx)\,dx$$
$$b_n = \frac{2}{\pi}\int_0^\pi g(x)\sin(nx)\,dx$$

Proof: With $f(x) = \sin(nx), g(x) = 0$, the solution is $f(x,t) = \cos(ct)\sin(nx)$. With $f(x) = 0, g(x) = \sin(nx)$, the solution is $f(x,t) = \frac{1}{c}\sin(ct)\sin(nx)$. For $f(x) = \sum_{n=1}^{\infty} a_n \sin(nx)$ and $g(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$, we get the formula by summing these two solutions.


VISUALIZATION. We can just plot the graph of the function $f(x,t)$ or plot the string for different times t.

$f(x,0)$ $f(x,1)$ $f(x,2)$ $f(x,3)$ $f(x,4)$

TO THE DERIVATION OF THE HEAT EQUATION. The temperature $f(x,t)$ is proportional to the kinetic energy at x. Divide the stick into n adjacent cells and assume that in each time step, a fraction of the particles moves randomly either to the right or to the left. If $f_i(t)$ is the **energy** of particles in cell i at time t, then the energy of particles at time $t+1$ is proportional to $(f_{i-1}(t) - 2f_i(t) + f_{i+1}(t))(t)$. This is a discrete version of the second derivative because $dx^2 f_{xx}(t,x) \sim (f(x+dx,t) - 2f(x,t) + f(x-dx,t))$.
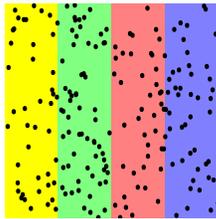
TO THE DERIVATION OF THE WAVE EQUATION. We can model a string by n discrete particles linked by strings. Assume that the particles can move up and down only. If $f_i(t)$ is the **height** of the particles, then the right particle pulls with a force $f_{i-1} - f_i$, the left particle with a force $f_{i-1} - f_i$. Again, $(f_{i-1}(t) - 2f_i(t) + f_{i+1})(t)$ which is a discrete version of the second derivative because $dx^2 f_{xx}(t,x) \sim (f(x+dx,t) - 2f(x,t) + f(x-dx,t))$.
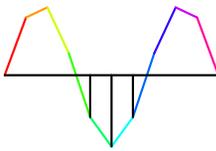
OVERVIEW: The heat and wave equation can be solved like ordinary differential equations:

Ordinary differential equations	Partial differential equations
$$\begin{aligned} x_t(t) &= Ax(t) \\ x_{tt}(t) &= Ax(t) \end{aligned}$$	$$\begin{aligned} f_t(t,x) &= f_{xx}(t,x) \\ f_{tt}(t,x) &= f_{xx}(t,x) \end{aligned}$$
Diagonalizing A leads for eigenvectors $\vec{v}$ $$Av = -c^2 v$$ to the differential equations $$\begin{aligned} v_t &= -c^2 v \\ v_{tt} &= -c^2 v \end{aligned}$$ which are solved by $$\begin{aligned} v(t) &= e^{-c^2 t}v(0) \\ v(t) &= v(0)\cos(ct) + v_t(0)\sin(ct)/c \end{aligned}$$	Diagonalizing $T = D^2$ with eigenfunctions $f(x) = \sin(nx)$ $$Tf = -n^2 f$$ leads to the differential equations $$\begin{aligned} f_t(x,t) &= -n^2 f(x,t) \\ f_{tt}(x,t) &= -n^2 f(x,t) \end{aligned}$$ which are solved by $$\begin{aligned} f(x,t) &= f(x,0)e^{-n^2 t} \\ f(x,t) &= f(x,0)\cos(nt) + f_t(x,0)\sin(nt)/n \end{aligned}$$

NOTATION:

f function on $[-\pi,\pi]$ smooth or piecewise smooth.
t time variable
x space variable
D the partial differential operator $Df(x) = f'(x) = d/dx f(x)$.
T linear transformation, like $Tf = D^2 f = f''$.
c speed of the wave.

$Tf = \lambda f$ Eigenvalue equation analogously to $Av = \lambda v$.
f_t partial derivative of $f(x,t)$ with respect to time t.
f_x partial derivative of $f(x,t)$ with respect to space x.
f_{xx} second partial derivative of f twice with respect to space x.
μ heat conductivity
$f(x) = -f(-x)$ odd function, has sin Fourier series

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1) Solve the heat equation $f_t = \mu f_{xx}$ on $[0,\pi]$ with the initial condition $f(x,0) = |\sin(3x)|$.

We want to see in exercises 2-4 how to deal with solutions to the heat equation, where the boundary values are not 0. Problems 2-4 belong together:

2) Verify that for any constants a, b the function $h(x,t) = (b-a)x/\pi + a$ is a solution to the heat equation.

3) Assume we have the problem to describe solutions $f(x,t)$ to the heat equations, where $f(0,t) = a$ and $f(\pi,t) = b$. Show that $f(x,t) - h(x,t)$ is a solution of the heat equation with $f(0,t) = 0$ and $f(\pi,t) = 0$.

4) Solve the heat equation with the initial condition $f(x,0) = f(x) = \sin(3x) + x/\pi$ and satisfying $f(0,t) = 0, f(\pi,t) = 1$ for all times t. This is a situation, when the stick is kept at constant but different temperatures on the both ends.

5) A piano string is fixed at the ends $x = 0$ and $x = \pi$ and initially undisturbed. The piano hammer induces an initial velocity $u_t(x,t) = g(x)$ onto the string, where $g(x) = \sin(2x)$ on the interval $[0,\pi/2]$ and $g(x) = 0$ on $[\pi/2,\pi]$. Find the motion of the string.