Math 21b Fall '97 Exam 1

1. True or false?

   (a) For any matrix A, $\mathrm{Im}\,(A) = \mathrm{Im}\,(\mathrm{rref}\,(A))$.

   (b) For any matrix A, $\dim\,(\mathrm{Im}\,(A)) = \mathrm{rank}\,(A)$.

   (c) If $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are any linearly dependent vectors in R^n, then $\vec{v}_3$ is a linear combination of $\vec{v}_1$ and $\vec{v}_2$.

   (d) There is a 3×6 matrix whose kernel is two-dimensional.

   (e) There is a 2×2 matrix A such that $A^2 = -I_2$.

2. Each of the spaces V_i below is equal to one (and only one) of the spaces W_j. Match the spaces.

$$V_1 = \mathrm{Im}\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad W_1 = \mathrm{Im}\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$V_2 = \mathrm{Im}\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \qquad W_2 = \mathrm{Im}\begin{bmatrix} 1 & 0 \\ -1 & -1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$V_3 = \mathrm{Ker}\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \qquad W_3 = \mathrm{Ker}\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$V_4 = \mathrm{Ker}\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \qquad W_4 = \mathrm{Ker}\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

$$V_5 = \mathrm{Span}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}\right) \qquad W_5 = \mathrm{Span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right)$$

3. Let $A = \begin{bmatrix} 0 & 1 & 3 \\ -1 & 0 & 0 \\ 0 & 2 & 5 \end{bmatrix}$.

   (a) Is A invertible? If so, find A^{-1}.

   (b) Find A^2.

4. Let A be a 2×2 matrix (not equal to I_2) representing a shear parallel to a line L in the plane. Find

   (a) $\mathrm{Ker}\,(A - I_2)$

   (b) $\mathrm{Im}\,(A - I_2)$

   (c) $(A - I_2)^2$

5. (a) Let A be a 3×3 matrix for which $\mathrm{Im}\,(A) = \mathrm{Span}\left(\begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}\right)$. What is $\mathrm{rank}\,(A)$? Give an example of such a matrix A.

   (b) Let B be a 3×3 matrix for which $\mathrm{Ker}\,(B) = \mathrm{Span}\left(\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}\right)$. What is $\mathrm{rank}\,(B)$? Give an example of such a matrix B.

   (c) Could you have chosen A and B so that $\mathrm{rank}\,(AB) = 2$? Briefly justify your answer.